Please use an extra sheet of paper for each problem 1, 2, and 3. Don't forget your name.

1 Knowledge (66 P.)

a. What does symmetry in physics mean (2)? How does one model symmetries (1)? What is the benefit of using symmetries (5)? Please provide three examples, specify the symmetry, provide the group by which it is modelled and describe a benefit for each case (9).

b. Define the mathematical term group (4)? What does Abelian mean (1)? What is the order of a finite group (1)?

c. What is a cyclic group (2)? List as many properties as you can find (3).

d. In connection with translational symmetry the following operator is used

$$\underset{\sim}{T}_a \;=\; \mathrm{e}^{-\frac{i a \underset{\sim}{p}}{\hbar}} \;, \quad a \in \mathbb{R} \;. \tag{1}$$

Explain why $\underset{\sim}{T}_a$ is unitary (1 P.). Provide a short derivation how $\underset{\sim}{T}_a$ acts on a state $|\,\phi\,\rangle$ (3 P.). Provide a short derivation to show what $\underset{\sim}{T}_a \underset{\sim}{x} \underset{\sim}{T}_a^\dagger$ is (3 P.). The operators $\underset{\sim}{T}_a$ together with the multiplication form a group. Which properties does this group possess (1 P.)?

e. We investigate a hydrogen atom whose energy levels can be labeled according to the irreps of $SO(3)$, i.e., by angular momentum quantum numbers l and m. Derive the selection rules for dipol transitions induced by the transition operator $\underset{\sim}{\vec{x}}$ from the Wigner-Eckart theorem (10)

f. A Cartesian tensor of rank two, $x_m x_n$, is made of the components of a position vector. This object can be reformulated so that the various parts transform like spherical tensors (or sperical harmonics) with $\ell = 0, 1, 2$. Why (3)? How does this decomposition look like in general (3)? Please provide this decomposition (7).

g. A Lie group is characterized by d real parameters. Define the Lie algebra (3). Which relation can be set up between the generators (3)? Mention at least one more property of your generators (1).

2 $O(2)$ (21 P.)

We consider the group $O(2)$. This group contains matrices of two types

$$A_1(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \;, \tag{2}$$

$$A_2(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \;. \tag{3}$$

a. Show that these matrices fulfill the special definition of $O(2)$ that $A^T A = \mathbf{1}$ (4).

b. What is the difference between the two types in terms of what they do (describe) and mathematically (6)?

c. Calculate the distance of both types to $\mathbf{1}$. Provide your definition of distance in general (6).

d. Determine the generators. How many do you get? Give a geometrical interpretation (3).

e. Matrix A_2 can be written as $A_1 M$ with some constant matrix M. Determine this matrix (2).

3 Spin rings (32 P.)

We study the Heisenberg model on a ring. More precisely, let there be $N = 6$ spins $s = 1/2$ interacting according to the Hamiltonian

$$\underset{\sim}{H} = J \sum_{i=1}^{N} \underset{\sim}{\vec{s}}_i \cdot \underset{\sim}{\vec{s}}_{i+1} \ , \tag{4}$$

with $\underset{\sim}{\vec{s}}_{N+1} \equiv \underset{\sim}{\vec{s}}_1$ (periodic boundary conditions, pbc).

a. Which symmetries does this Hamiltonian possess? Give reason. By which groups are these symmetries modelled (6)?

b. Provide the product basis for the above Hilbert space and explain. What is its dimension (4)?

c. How can a product representation $D^{(S_1)} \otimes D^{(S_2)}$ of two $SU(2)$ irreps be decomposed into irreps in general? Provide the final formula and explain the quantities (5).

d. What are the possible total spin quantum numbers of the above spin ring of $N = 6$ spins $s = 1/2$? Give reason and also derive their multiplicities with the help of a coupling scheme of your choice. Check the dimension (5).

e. Define the group of translations on the above ring and provide a general formula for basis states of the invariant subspaces belonging to the irreps by acting with a projector on states of the product basis (5).

f. We now condider a smaller system of $N = 4$ spins $s = 1/2$ and look at the subspace $\mathcal{H}(M)$ with total magnetic quantum number $M = 0$. Write down the product basis in this subspace and determine the dimension of this subspace. Decompose this subspace into subspaces belonging to the irreps of the translational symmetry and provide the basis states in these subspaces. In the end, check the dimensions (7).

You can gather 119 points and obtain the following marks

- $0 \leq P \leq 50 \Rightarrow 5.0$
- $51 \leq P \leq 55 \Rightarrow 4.0$
- $56 \leq P \leq 60 \Rightarrow 3.7$
- $61 \leq P \leq 65 \Rightarrow 3.3$
- $66 \leq P \leq 70 \Rightarrow 3.0$
- $71 \leq P \leq 75 \Rightarrow 2.7$
- $76 \leq P \leq 80 \Rightarrow 2.3$
- $81 \leq P \leq 85 \Rightarrow 2.0$
- $86 \leq P \leq 90 \Rightarrow 1.7$
- $91 \leq P \leq 95 \Rightarrow 1.3$
- $96 \leq P \leq \infty \Rightarrow 1.0$