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10 Problem sheet

10.1 IN CLASS (finish at home): Block structure of Hamiltonian, neutron scattering

Consider the product Hilbert space for three spins $s_i = 1$ and the following Heisenberg Hamiltonian

$$\underset{\sim}{H} = J_1 \underset{\sim}{\vec{s}}_1 \cdot \underset{\sim}{\vec{s}}_2 + J_2 \left(\underset{\sim}{\vec{s}}_2 \cdot \underset{\sim}{\vec{s}}_3 + \underset{\sim}{\vec{s}}_3 \cdot \underset{\sim}{\vec{s}}_1 \right) \ . \tag{17}$$

a. Explain, why we call this Hamiltonian rotationally invariant. How can we express this property mathematically?

b. Repeat how we decompose the product representation into irreps

$$D^{(s_1)} \otimes D^{(s_2)} \otimes D^{(s_3)} \quad = \quad \bigoplus_S n_S D^{(S)} \tag{18}$$

and provide the total spin quantum numbers as well as their multiplicities.

c. Write down the coupling scheme you used both as a tree and as ket basis states.

d. How could one show that these states form an ONB?

e. Make a graphical sketch that explains into how many blocks the Hamiltonian matrix decays. Label these blocks and explain.

f. Now calculate the entries of all blocks. Recall the properties of the basis states of your coupling scheme. Of which operators are these states eigenstates? Although you can choose every coupling scheme you like, there is a more practical coupling scheme in this case.

g. Determine the energy eigenvalues.

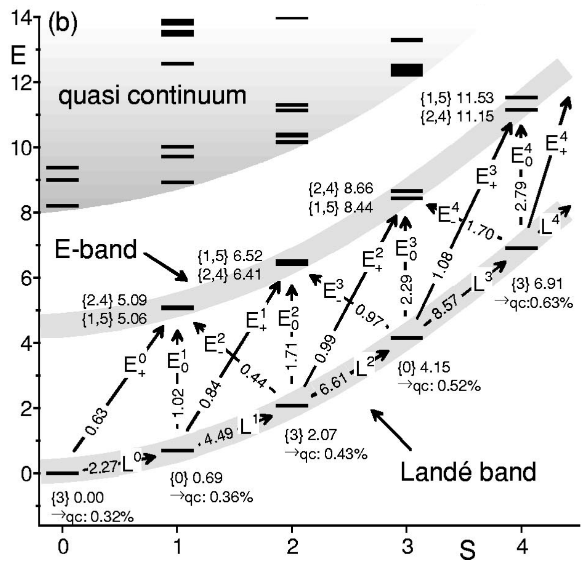

Figure 3: Low-energy part of the energy spectrum of a spin ring with $N = 6$ spins with $s_i = 5/2$. The arrows show some transitions, many transitions do not exist (jargon "are forbidden"), compare Oliver Waldmann, Spin dynamics of finite antiferromagnetic Heisenberg spin rings, Phys. Rev. B 65, 024424 (2001).

h. Inelastic neutron scattering (INS) can be used to determine the energy differences between eigenenergies. In an INS spectrum peaks occur at such energies called resonance energies (or frequencies). However, not all such differences are measurable since the transition matrix element between initial and final state might be zero. DUE TO SYMMETRY.

   The transition operator is proportional to the spin vector operators in the system (or a linear combination of them). Please explain which states can in principle be reached from an eigenstate of a rotationally invariant Hamiltonian if the initial state is $\mid SM;\text{quantum numbes of coupling scheme}\,\rangle$ and we consider $s_q^{(\alpha)}$ as the transition operator.

i. We did not speak much about the M quantum number, nor much about q in $s_q^{(\alpha)}$. Could you make a statement about them? Consider the Clebsch-Gordan coefficients for your argument.

j. Read Section II of the paper by Oliver Waldmann. It should ring in your head. Many times. Congratulate yourself for attending this course. ;-)

Merry Christmas & Happy New Year