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7 Problem sheet

7.1 IN CLASS: some invariant subspaces for C_{3v}

We consider the group C_{3v}.

a. Following the example presented in the lecture consider three spins $s_i = 1$ and the subspace of total magnetic quantum number $M = 1$. Write down the product basis and the dimension of this space.

b. Write down the projectors onto the irreps of C_{3v} according to the lecture.

c. Construct the invariant subspaces by applying the projectors to the states of the product basis. Check for linear independence and reduce the sets of obtained vectors accordingly.

7.2 AT HOME: literature

Group tables, irreducible representations as well as characters are tabulated in some literature.

a. Look up a book or source of group representations and characters. Provide the reference of the book/source.

b. Look at your matrikel number and take the last digit x. If $x = 0$, take $x = 5$. Find a non-Abelian group of order x or $2x$ and provide the details about this group, in particular the character table.

7.3 AT HOME: Prove properties of projector on invariant subspaces

Prove the following relations:

a. $P^{(\alpha)}_{lk} P^{(\beta)}_{l'k'} = \delta_{\alpha\beta}\delta_{kl'} P^{(\alpha)}_{lk'}$

b. $\left(P^{(\alpha)}_{lk}\right)^{\dagger} = P^{(\alpha)}_{kl}$

c. $P^{(\alpha)} = \sum_{k=1}^{d_\alpha} P^{(\alpha)}_{kk}$

You can use that the representations are unitary.

7.4 AT HOME: Rotations in spin (angular momentum) space

We will see in the lecture in more depth later that the spin is the generator of rotations as is the momentum operator for translations.
Make a hypothesis what the following relation yields, maybe in words, then calculate

$$\exp\{-i\frac{\alpha \underset{\sim}{s}_z}{\hbar}\} \; \underset{\sim}{\vec{s}} \; \exp\{i\frac{\alpha \underset{\sim}{s}_z}{\hbar}\} \tag{8}$$

for a single spin. For a derivation you may use the lemma by Hadamard (German: Liesche Entwicklungsformel)

$$\exp\{\underset{\sim}{A}\} \; \underset{\sim}{B} \; \exp\{-\underset{\sim}{A}\} \;\; = \;\; \sum_{m=0}^{\infty} \frac{1}{m!}[\underset{\sim}{A}, \underset{\sim}{B}]_m \; , \tag{9}$$

with

$$[\underset{\sim}{A}, \underset{\sim}{B}]_m \;\; = \;\; [\underset{\sim}{A}, [\underset{\sim}{A}, \underset{\sim}{B}]_{m-1}] \tag{10}$$

and

$$[\underset{\sim}{A}, \underset{\sim}{B}]_0 \;\; = \;\; \underset{\sim}{B} \; . \tag{11}$$

Decompose the spin operator into its components and consider (8) for each component. Sort the terms cleverly and use the series representation of known functions.

Advanced problem: The total spin operator is the generator of a collective rotation of all spins. Why?
Write down how it acts and consider the appropriate commutation relations between spins.