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6 Problem sheet

6.1 IN CLASS (finish at home): irreps of C_{3v}

We consider the group C_{3v}.

a. Repeat the group table, the cosets and the conjugacy classes.

b. This group has got three irreps, see table (Trebbin, 2011).

$$\begin{array}{lll}
\boldsymbol{D}^{(1)}(e) = 1 & \boldsymbol{D}^{(1)}(c_3) = 1 & \boldsymbol{D}^{(1)}(c_3^2) = 1 \\
\boldsymbol{D}^{(1)}(\sigma) = 1 & \boldsymbol{D}^{(1)}(\sigma') = 1 & \boldsymbol{D}^{(1)}(\sigma'') = 1 \\
\boldsymbol{D}^{(2)}(e) = 1 & \boldsymbol{D}^{(2)}(c_3) = 1 & \boldsymbol{D}^{(2)}(c_3^2) = 1 \\
\boldsymbol{D}^{(2)}(\sigma) = -1 & \boldsymbol{D}^{(2)}(\sigma') = -1 & \boldsymbol{D}^{(2)}(\sigma'') = -1
\end{array}$$

Figure 1: C_{3v}– two one-dimensional irreps.

$$\boldsymbol{D}^{(3)}(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{D}^{(3)}(c_3) = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

$$\boldsymbol{D}^{(3)}(c_3^2) = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} \qquad \boldsymbol{D}^{(3)}(\sigma) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{D}^{(3)}(\sigma') = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \qquad \boldsymbol{D}^{(3)}(\sigma'') = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$$

Figure 2: C_{3v}– one two-dimensional irrep.

Check the theorem of Burnside about the relation between dimensions of the irreps and the order of the group.

c. Check the orthogonality theorem between these three irreps.

d. Set up the character table for C_{3v} and check the orthogonality relation for characters.

6.2 AT HOME: Cyclic groups

Consider a cyclic group such as C_N.

a. How many conjugacy classes does this group possess?

b. What are the irreps and what are their characters?

6.3 AT HOME: Entanglement, fermions, and bosons

a. Let V be a vector space spanned by the basis vectors $|+\rangle$ and $|-\rangle$. Show that the vector $|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle$ cannot be written as a direct product of any two vectors from V. This is at the heart of entanglement.

b. Let A be an automorphism of a vector space V and $A \otimes A$ a corresponding automorphism of $V \otimes V$. Show that $A \otimes A$ leaves the symmetric and antisymmetric subspaces of $V \otimes V$ invariant. To this end, first define these subspaces. Is it clear that they are subspaces?

6.4 AT HOME: A small advanced challenge

Consider the product Hilbert space for three spins $s_i = 1/2$ and the following Heisenberg Hamiltonian

$$\underset{\sim}{H} = J\left(\underset{\sim}{\vec{s}}_1 \cdot \underset{\sim}{\vec{s}}_2 + \underset{\sim}{\vec{s}}_2 \cdot \underset{\sim}{\vec{s}}_3 + \underset{\sim}{\vec{s}}_3 \cdot \underset{\sim}{\vec{s}}_1\right) \ . \tag{7}$$

a. What are symmetries of this Hamiltonian? How would one check these symmetries?

b. Write down the product basis of the product Hilbert space.

c. Can you deduce invariant Hilbert subspaces according to the symmetries? Please show.

d. Do you manage to turn the Hilbert space into a direct sum of one dimensional spaces only? Please demonstrate.