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5 Problem sheet

5.1 IN CLASS: Representations of Z_4

We consider the group $Z_4 = \{e, a, a^2, a^3\}$.

a. We will get to know the regular representation in the lecture. For the moment and for Z_4 we define the action of the group elements on the basis vectors $\{e_1, e_2, e_3, e_4\}$ of a four-dimensional vector space as $ae_i = e_{a(i)}$ using the relation to the cyclic permutations of the symmetric group.

   Write down the four (4×4)-matrices.

b. There is also a representation by (3×3)-matrices:

$$D(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \ . \tag{5}$$

   Construct the representations of the other three group elements.

c. In a two-dimensional vectore space we could employ

$$D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{6}$$

   Construct the representations of the other three group elements.

d. Which of the above are faithful representations?

5.2 AT HOME: Coming back to translational symmetry

For the one-dimensional chain with periodic boundary conditions we employed the translation operator whose eigenbasis allowed us to construct a representation in one-particle space that was already diagonal. We will soon see in the lecture that this corresponds to irreducible representations of the group Z_N (or C_N). With this approach we could obtain the spin waves of a one-dimensional ferromagnetic.
The power of math lies in the fact that the solution is only related to the mathematical structure of the problem not to the physical meaning. We can therefore apply all we learned to other, but mathematically similar problems.
Let's look at the one-dimensional oscillator chain.

a. Make a sketch of the one-dimensional oscillator chain. All masses as the same, and all spring constants are the same. Like a one-dimensional structure in a solid there would be ends that are, e.g., attached to walls. Again, we assume that we can safely approximate the situation by periodic boundary conditions.

   Write down the coupled equations of motion for this system and bring it into matrix-vector form. How does the coupling matrix look like?

b. The coupling matrix has the same mathematical form as the Heisenberg Hamiltonian in one-magnon space expressed with the help of the product basis. You solved this already. This matrix can be analytically diagonalized by a basis transform to the eigenbasis of the translation “operator” although we don’t work in Hilbert space now, but we know what a translation would do.

   Read the provided script. Don’t be scared of the German. If you have a question about some comment, please drop me a line.

   You should be able to explain the story in class.

c. Check that the $\vec{e}_k$ on page 6-2-E are eigenvectors of T. Do it in general.

   Here T is used both as a map and as its representation. This is to train you since such a confusions also appears in many books.

d. How can you make Kronecker symbols with sums of roots of unity?

e. Why is the last line on page 6-2-E ingeneous?

Die endgültige Musterlösung

mit hilfreichen Eingaben von Jannis

$$\ddot{\vec{q}} = -\omega_0^2 \overleftrightarrow{K} \cdot \vec{q} \qquad \vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} \text{ N Koord.}$$

$$\overleftrightarrow{K} = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

Was bedeutet $\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$?

↳ Es ex. eine ONB $\{\vec{b}_1, \dots, \vec{b}_N\}$ mit $q_i = \vec{b}_i \cdot \vec{q}$.

Problem: $\vec{q}$ bezeichnet den Vektor und seine Darstellung bzgl. der ONB $\{\vec{b}_i\}$!

Entkopplung der DGL durch Basistransformation auf Eigenbasis von $\overleftrightarrow{K}$!
$\overleftrightarrow{K}$ ist reell symmetrisch und hat deshalb reelle Eigenwerte und orthogonale Eigenvektoren (zu unterschiedlichen EW).

Was sind die Einträge von $\overleftrightarrow{K}$?

$K_{ij} = \vec{b}_i \cdot \overleftrightarrow{K} \cdot \vec{b}_j$ Anmerk.: $\vec{b}_i^T \cdot \overleftrightarrow{K} \cdot \vec{b}_j$ weggelassen

$\overleftrightarrow{K}$ ist also auch gleichzeitig lineare Abbildung und deren Darstellung.

→

6 – 2 – B

Eigenbasis von $\overleftrightarrow{K}$ seien $\vec{e}_k$, $k = 1, \ldots, N$

$$\overleftrightarrow{K} \cdot \vec{e}_k = K^D_{kk}\, \vec{e}_k$$

$\overleftrightarrow{K}^D$ ist eine Diagonalmatrix

K^D_{kk} ist der k-te Eigenwert

$$\ddot{\vec{q}} = -\omega_0^2\, \overleftrightarrow{K} \cdot \vec{q} \qquad \mathbb{1} = \sum_k \vec{e}_k \otimes \vec{e}_k$$

$$= -\omega_0^2 \sum_k \overleftrightarrow{K}\, \vec{e}_k \otimes \vec{e}_k \cdot \vec{q} = -\omega_0^2 \sum_k \overleftrightarrow{K} \cdot \vec{e}_k\; \vec{e}_k \cdot \vec{q} \quad \Big| \cdot \vec{e}_\ell$$

$$\vec{e}_\ell \cdot \ddot{\vec{q}} = -\omega_0^2 \sum_k \underbrace{\vec{e}_\ell \cdot \underbrace{\overleftrightarrow{K} \cdot \vec{e}_k}_{K^D_{kk}\, \vec{e}_k}}_{\delta_{\ell k}}\; \vec{e}_k \cdot \vec{q}$$

$$= -\omega_0^2\, K^D_{\ell\ell}\; \vec{e}_\ell \cdot \vec{q}$$

$\vec{e}_\ell \cdot \vec{q} = Q_\ell$ ist die Darstellung des Vektors $\vec{q}$ bzgl. der Eigenbasis $\{\vec{e}_k\}$

$$\Rightarrow \ddot{Q}_\ell = -\underbrace{\omega_0^2\, K^D_{\ell\ell}}_{\Omega_\ell^2}\, Q_\ell \,, \quad \ell = 1, \ldots, N$$

$$\Rightarrow Q_\ell(t) = Q_\ell^A \cdot \cos \Omega_\ell t + Q_\ell^B \sin \Omega_\ell t \quad //$$

AB: $\vec{q}(0)$ und $\dot{\vec{q}}(0)$ kann umgerechnet werden mittels

$$Q_\ell(0) = \vec{e}_\ell \cdot \vec{q}(0)\,, \quad \dot{Q}_\ell(0) = \vec{e}_\ell \cdot \dot{\vec{q}}(0) \quad //$$

6-2-c

Wie verhalten sich die Basen zueinander?

i. $\vec{b}_i = \sum_k \vec{e}_k \otimes \vec{e}_k \cdot \vec{b}_i = \sum_k \vec{e}_k \underbrace{\vec{e}_k \cdot \vec{b}_i}$

Darstellung der $\vec{b}_i$ bzgl. der $\vec{e}_k$

ii. $\vec{e}_k = \sum_i \vec{b}_i \otimes \vec{b}_i \cdot \vec{e}_k = \sum_i \vec{b}_i \underbrace{\vec{b}_i \cdot \vec{e}_k}$

Darstellung der $\vec{e}_k$ bzgl. der $\vec{b}_i$

Damit auch

$$Q_k = \vec{e}_k \cdot \vec{q} = \sum_j \vec{e}_k \cdot \vec{b}_j \otimes \vec{b}_j \cdot \vec{q} = \sum \vec{e}_k \cdot \vec{b}_j \; \vec{b}_j \cdot \vec{q}$$

$$= \sum_j \vec{e}_k \cdot \vec{b}_j \; q_j \;//$$

$$q_j = \vec{b}_j \cdot \vec{q} = \sum_k \vec{b}_j \, \vec{e}_k \otimes \vec{e}_k \, \vec{q} = \sum_k \vec{b}_j \cdot \vec{e}_k \; \vec{e}_k \cdot \vec{q}$$

$$= \sum_k \vec{b}_j \cdot \vec{e}_k \; Q_k \;//$$

$\Rightarrow \vec{Q} = \overleftarrow{U} \vec{q}, \quad Q_k = \sum_j U_{kj} q_j$

mit periodischen Randbedingungen

$$\vec{e}_k \cdot b_j = \frac{1}{\sqrt{N}} e^{+i \frac{2\pi k \cdot j}{N}}$$

$j = 1, \dots, N; \; k = 0, \dots, N-1$

Konvention

$$K^{D}_{kk} = \underset{\uparrow\,\mathbb{1}}{\vec{e}_k \cdot} \overleftarrow{K} \underset{\uparrow\,\mathbb{1}}{\cdot} \vec{e}_k = \sum_{j_1, j_2} \vec{e}_k \cdot \vec{b}_{j_1} \otimes \vec{b}_{j_1} \cdot \overleftarrow{K} \cdot \vec{b}_{j_2} \otimes \vec{b}_{j_2} \cdot \vec{e}_k$$

$$= \sum_{j_1, j_2} \vec{e}_k \cdot \vec{b}_{j_1} \underbrace{\vec{b}_{j_1} \cdot \overleftarrow{K} \cdot \vec{b}_{j_2}}_{K_{j_1 j_2}} \underbrace{\vec{b}_{j_2} \cdot \vec{e}_k}_{= (\vec{e}_k \cdot \vec{b}_{j_2})^{+}}$$

6 – 2 – D

$$\Rightarrow K^D_{kk} = \frac{1}{N} \sum_{j_1 j_2} e^{i \frac{2\pi k}{N} j_1} K_{j_1 j_2} e^{-i \frac{2\pi k}{N} j_2}$$

mit $K_{j_1 j_2} = 2\delta_{j_1 j_2} - 1\delta_{j_2 j_1+1} - 1\delta_{j_2 j_1-1}$

$N+1 \hat{=} 1, \quad 1-1 \hat{=} N$

$$\Rightarrow K^D_{kk} = \frac{1}{N}\left\{ 2 \cdot N - 1 \cdot N \cdot e^{i\frac{2\pi k}{N}} - 1 \cdot N e^{-i\frac{2\pi k}{N}} \right\}$$

(*) $$= 2 - 2\cos\frac{2\pi k}{N}, \quad k = 0, \dots, N-1$$

Bsp.: $N=2$: $K^D_{00} = 0$, $K^D_{11} = 4$ ✓

$N=3$: $K^D_{00} = 0$, $K^D_{11} = 3$, $K^D_{22} = 3$ ✓

Warum sind die $K^D_{k_1 k_2} = 0$?

$$K^D_{k_1 k_2} = \frac{1}{N}\left\{ 2\sum_{j_1 j_2} e^{i\frac{2\pi}{N}(k_1 - k_2) j_1} \delta_{j_1 j_2} - 1 \sum_{j_1 j_2} e^{i\frac{2\pi}{N}(k_1-k_2) j_1} \cdot e^{i\frac{2\pi}{N} k_2} \delta_{j_2 j_1+1} - 1 \dots \right\}$$

$$0 \quad \frac{1}{N}\sum_{j=1}^{N} e^{i\frac{2\pi}{N}(k_1 - k_2) j} = \delta_{k_1 k_2}$$

000

$$\Rightarrow K^D_{k_1 k_2} = 0 \quad \text{für } k_1 \neq k_2$$

$\rightarrow \hat{K}^D$ in der Tat diagonal ?

(*) $2\left(1 - \cos\frac{2\pi k}{N}\right) = 4\sin^2\frac{\pi k}{N}$ //

mit $\cos\left(\frac{\pi k}{N} + \frac{\pi k}{N}\right) = \cos^2\frac{\pi k}{N} - \sin^2\frac{\pi k}{N} = 1 - 2\sin^2\frac{\pi k}{N}$

jetzt noch einmal der Translationsoperator

Definiere: $\overleftarrow{T}\vec{b}_i = \vec{b}_{i+1}$, $N+1 \hat{=} 1$

$\Rightarrow \vec{q} = \sum_j q_j \vec{b}_j$, $\overleftarrow{T}\vec{q} = \sum_j q_j \overleftarrow{T}\vec{b}_j = \sum_j q_j \vec{b}_{j+1}$

$\hat{=} \quad \overleftarrow{T}\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} q_N \\ q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}$?

? Konstruiere: $\vec{e}_k = \frac{1}{\sqrt{N}} \sum_{\nu=0}^{N-1} \left(e^{-i\frac{2\pi k}{N}}\overleftarrow{T}\right)^\nu \vec{b}_1$ ← Eins

mit $k = 0, 1, \ldots, N-1$

Bsp.: $\vec{e}_0 = \frac{1}{\sqrt{N}}\left(\vec{b}_1 + \vec{b}_2 + \ldots + \vec{b}_N\right)$

$\vec{e}_1 = \frac{1}{\sqrt{N}}\left(\vec{b}_1 + e^{-i\frac{2\pi}{N}}\vec{b}_2 + e^{-\frac{4\pi}{N}}\vec{b}_3 \cdots\right)$

...

Lösung: 1. $\vec{e}_k$ ist Eigenvektor von $\overleftarrow{T}$?

2. $\vec{e}_k$ diagonalisiert $\overleftarrow{K}$?

(siehe 6-2-D)

Hier jetzt die spezielle Lsg.:

$\overleftarrow{K} = 2\underline{\mathbb{1}} + T - T^{-1}$