Flatband makes the wave go round

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Persistent oscillations are a hallmark of non-ergodic time evolution. While time-crystalline behavior results from, e.g., many-body localization, here we show that ever-revolving solitary waves emerge in flatband Heisenberg quantum spin systems.

Introduction.—Ever-lasting oscillations are a fascinating phenomenon that got a new twist with the advent of time-crystals [1–4], where a quantum system exhibits oscillatory behaviour of some of its observables. The topic is still so new that every now and then hot debates try to settle the subject, compare [5–7], but often do not make it into publications since contrary to the time crystal the field moves on. Some systematic of time-crystalline behavior is provided in Figure 8 of [8].

The example we want to discuss in the following belongs to the class of non-driven closed Hamiltonian systems, and thus has got some relationship with quantum scars [9–13] as well as Hilbert space fragmentation [14–16]. The appearance of quantum scars, Hilbert space fragmentation or time crystals signals non-ergodic/non-thermalizing behavior that contradicts our expectation of thermalization that nowadays is the natural expectation for generic quantum systems even when closed and under unitary time evolution [17–27]. And in order not to interfere with the still evolving notion of time crystals we prefer to qualify the persistant oscillatory dynamics discussed in this article as sufficiently non-trivial (compared to trivial Bloch oscillations or single-spin Larmor precessions). Moreover, it is related to the observation of solitary waves in quantum spin systems [28].

Magnetic solitons have been detected experimentally in several magnetic systems [31–35] for instance as domain-wall like or envelope solitons. From a theoretical point of view magnetic solitons are solutions of non-linear differential equations as for instance the cubic Schrödinger equation [36, 37] usually as the result of an approximation of the time-dependent Schrödinger equation, which is linear. For instance, the cubic Schrödinger equation arises when quantum spins are replaced by a classical spin density [38].

As discussed already in Ref. [28], the *linear* time-dependent Schrödinger equation allows for solutions that move along or across some translationally symmetric quantum spin system with frozen shape. Such states shall be called solitary waves. More precisely, we want to call $\mid \Psi_s \,\rangle$ a solitary wave if there exists a minimal time $\tau > 0$ for which the time evolution equals (up to a global phase) the shift by one unit cell of the spin system [28]. A state $\mid \Psi_s \,\rangle$ that is a superposition of energy eigenstates $\mid k, E = \alpha k + E_0 \,\rangle, k = 0, 1, 2, \ldots$ with a linear dispersion relation would behave as a solitary wave and move on forever. With periodic boundary conditions as for instance naturally given in spin rings this would lead to everlasting revolutions around the closed structure.

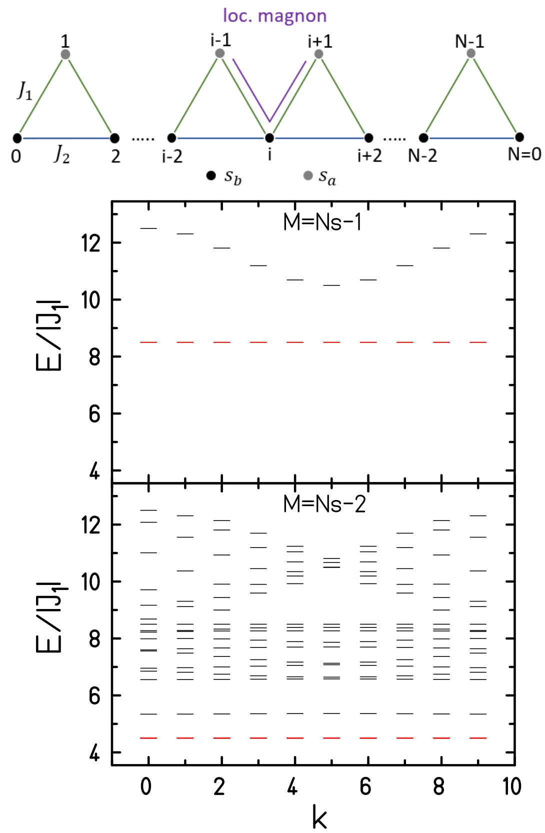

Figure 1. Top: Structure of the delta chain with apical spins s_a and basal spins s_b as well as exchange interactions J_1 and J_2. The spins are numbered $0, 1, \ldots, N-1$. Periodic boundary conditions are applied, i.e. $N \equiv 0$. An independent localized one-magnon state is highlighted that extends over three neighboring sites as indicated [29]. Bottom: Energy eigenvalues for $N = 20$ and $s_a = s_b = \frac{1}{2}$ for one-magnon ($M = Ns-1$) and two-magnon space ($M = Ns - 2$. The momentum quantum number k (wave number) runs from 0 to $N/2-1$, compare [30].

While Ref. [28] speculated that in a (dense) spectrum sufficiently many eigenstates with linear dispersion relation should exist, which typically is not the case, flatband systems give rise to *perfectly* linear dispersion relations even in non-dense spectra when combining appropriate multimagnon states [29, 39–45]. In order to demonstrate the approach as well as the resulting dynamics we choose the one-dimensional delta chain in the Heisenberg model with spins $s = 1/2$ which exhibits flat bands in serveral multimagnon subspaces for a ratio of the two defining exchange interactions of $J_2/J_1 = 1/2$, compare Fig. 1. We find it remarkable that flatband spin systems such as the delta chain thus give rise to two rather different phenomena: disorder-free localization with zero group velocity [11, 30] as well as solitary dynamics demonstrated in this paper.

Essential properties of flatband systems.—The antiferromagnetic delta chain, also termed sawtooth chain, is shown in Fig. 1 (top). It is modelled by the Heisenberg model with periodic boundary conditions

$$\underset{\sim}{H} = -2J_1 \sum_{i=0}^{N-1} \underset{\sim}{\vec{s}}_i \cdot \underset{\sim}{\vec{s}}_{i+1} - 2J_2 \sum_{i=0}^{\frac{N}{2}-1} \underset{\sim}{\vec{s}}_{2i} \cdot \underset{\sim}{\vec{s}}_{2i+2} \ . \tag{1}$$

$\underset{\sim}{\vec{s}}_i$ denotes the spin vector operator at site i, and $J_1 < 0$ as well as $J_2 < 0$ are antiferromagnetic exchange interactions. The unit cell contains two spins which gives rise to momentum quantum numbers $k = 0, 1, \dots, N/2 - 1$. Overall, the eigenstates can be organized according to the present symmetries and labeled with total spin S, total magnetic quantum number M, and momentum quantum number (wave number) k.

In one-magnon space two energy bands appear of which one is flat for $J_2/J_1 = 1/2$, see Fig. 1 (center). This property is equivalent to the existence of localized independent one-magnon states (sometimes also termed "compact localized states" [45–47]) of which one is shown in Fig. 1 (top). These states are not only eigenstates of the Hamiltonian in one-magnon space, but also ground states since they are given by Fourier transforms of the ground-state flat band [29]. Out of localized independent one-magnon states one can construct n-magnon states that are also eigen- and groundstates of the Hamiltonian in their respective n-magnon spaces up to the maximum possible number of localized independent magnons [29, 48], compare 2-magnon space in Fig. 1 (bottom). This leads to a strict linear dispersion between magnetic quantum number M and ground state energy of the $(Ns - M)$-magnon space.

The desired linear dispersion relation between E and k is then obtained by picking appropriate eigenstates $|\, M, k, \alpha \,\rangle$ from the respective degenerate ground state manifold. α serves as a label to enumerate the levels with a certain M and k. To be specific,

$$\begin{aligned} |\, \Psi_s \,\rangle =& c_0 \,|\, M = Ns, k = 0 \,\rangle \\ &+ c_1 \,|\, M = Ns - 1, k = 1, \alpha_1 \,\rangle \\ &+ c_2 \,|\, M = Ns - 2, k = 2, \alpha_2 \,\rangle \dots \ , \end{aligned} \tag{2}$$

where the first state is the magnon vacuum, the second state a $(k = 1)$-eigenstate from the flat ground state band in one-magnon space, the third a $(k = 2)$-eigenstate from the flat ground state band in two-magnon space, and so on. We consider a superposition of more than two eigenstates as non-trivial because it is unlikely for generic systems that more than two eigenstates fulfill a linear dispersion relation *exactly*.

In general, with $\underset{\sim}{U}$ being the time-evolution operator and $\underset{\sim}{T}$ the operator that translates (shifts) by one unit cell, solitary waves $|\, \Psi_s \,\rangle$ fulfill

$$\underset{\sim}{U}(\tau) \,|\, \Psi_s \,\rangle = e^{-i\phi_0} \underset{\sim}{T}^{\pm} \,|\, \Psi_s \,\rangle \tag{3}$$

for a certain discrete time τ (up to a global phase). Inserting decomposition (2) yields

$$E_\mu \tau / \hbar = \pm \frac{4\pi k_\mu}{N} + 2\pi m_\mu + \phi_0 \ , \ m_\mu \in \mathbb{Z} \tag{4}$$

mentioned above with some arbitrary constants due to properties of the complex unit circle. This results in a minimal τ of

$$\tau = \frac{\Delta k 4\pi\hbar}{\Delta E N} \ , \tag{5}$$

showing $\hbar$ explicitly for convenience [28].

To some extend the solitary waves can be shaped depending on the number, kind, and amplitude of its Fourier components, compare (2), since the flat band states are often degenerate.

Numerical example.—We looked at a delta chain with $N = 20$ and $s = 1/2$ at the flatband point $J_2/J_1 = 1/2$. Figure 2 shows the time evolution of the expectation value of individual operators $\underset{\sim}{s}_\ell^x$

$$\langle\, \underset{\sim}{s}_\ell^x \,\rangle = \langle\, \Psi_s \,|\, \underset{\sim}{s}_\ell^x \,|\, \Psi_s \,\rangle \tag{6}$$

as a function of time for a superposition of four energy eigenstates with $k = 0, 1, 2, 3$. One recognises two features: (1) every individual spin expectation value oscillates permanently, and (2) this oscillation has got an offset (of size τ) with respect to the neighboring unit cell. In total, the picture shows a wave that travels around the delta-chain ring.

A measure for the stability of the solitary wave is the overlap of the shifted state with the time evolved one (related to the Jozsa fidelity [49] after one revolution around the ring)

$$\eta(t) = \langle\, \Psi_s \,|\, \underset{\sim}{T}^{-1} \underset{\sim}{U}(t) \,|\, \Psi_s \,\rangle \ , \tau/2 \le t < 3\tau/2 \ . \tag{7}$$

We define $\eta(t)$ piece-wise and restart the procedure for the next interval accordingly. For a perfect solitary wave the absolute value of $\eta(n\tau)$, $n \in \mathbb{Z}$, is equal to 1. This is seen in Fig. 3 when looking at the blue curve that shows the case of a perfect solitary wave $|\, \Psi_0 \,\rangle = |\, \Psi_s \,\rangle$. If some random component is added to a perfect solitary wave, resulting in $|\, \Psi_1 \,\rangle$, $|\eta(t)|$ in general will not return

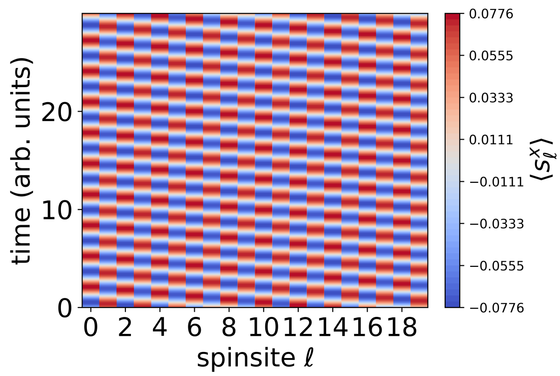

Figure 2. Time evolution of the expectation values $\langle\, \underset{\sim}{s}^x_\ell \,\rangle$ in a delta chain with $N = 20$ spins. The initial state is a superposition of four eigenstates according to (2). The characteristic time τ can be deduced from the shift by one unit cell (of two neighboring spins).

to its initial value. However, $|\eta(t)|$ periodically returns to some smaller value since the contribution of the solitary wave develops independently of the remainder for the linear Schrödinger equation. The random component might even equilibrate, i.e., smear out around the ring while the solitary-wave contribution still runs unperturbed (green curve), see discussion in [30]. If the Hamiltonian is slightly off the flatband scenario, i.e., possesses only dispersive bands, an approximate solitary wave $|\,\Psi_2\,\rangle$ slowly looses recurrence, and $|\eta(t)|$ decays while still oscillating (red curve in Fig. 3).

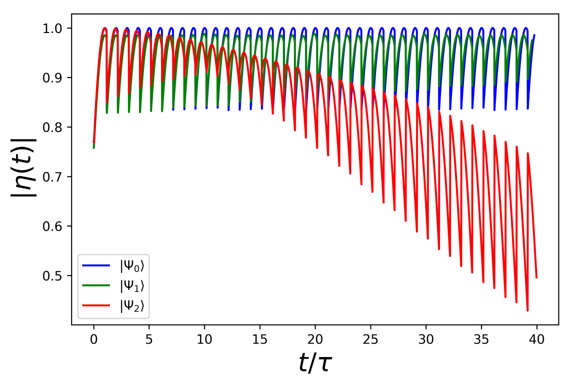

Figure 3. Overlap $\eta(t)$ of the shifted state with the time evolved state for a perfect solitary wave $|\,\Psi_0\,\rangle$ (blue), a solitary wave perturbed with a random admixture $|\,\Psi_1\,\rangle$ (green) and an approximate solitary wave in a system with a slightly dispersive band $|\,\Psi_2\,\rangle$ (red). For the latter case an approximate, effective τ was used as unit of time.

Discussion and conclusions.—In this paper, we demonstrated that certain carefully prepared initial states of flatband systems give rise to permanent oscillations and solitary wave behavior. We would like to remind the reader that a magnetic field is not involved. Quantum spin systems with flat bands such as the discussed delta chain, the kagome lattice, the square-kagome lattice, the pyrochlore lattice and several other frustrated flatband systems thus do not only show very exciting magnetic properties like spin-liquid behavior, magnetization plateaus and jumps, they also provide examples of non-generic, non-ergodic behavior expressed for instance in disorder-free localization with zero group velocity [11, 30] as well as persistent motion of solitary waves.

Of course, the peculiar dynamics is related to fine-tuned Hamiltonians and sometimes also fine-tuned initial states. Away from the flatband scenario, strictly permanent oscillations will not occur, however, depending on the strength of the dispersion they might be very long-lived.

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