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Problem sheet 13

13.1 Quadupolar order

Thermodynamic phases are characterized by an order parameter. During a phase transition the order parameter changes (abruptly). For a liquid-gas transition the density is an order parameter, which is scalar.
A ferromagnetic phase can be described by a vectorial order parameter, which is the magnetization. This phase, by the way, breaks time reversal invariance, since time reversal turns an angular momentum operator into its negative.
In recent times exotic quantum phases have gathered some interest. Among those are phases whose order parameter is given by spherical tensors of rank 2. The chapter *Spin Nematic Phases in Quantum Spin Systems* by Karlo Penc and Andreas Läuchli in the book *Introduction to Frustrated Magnetism* by Claudine Lacroix, Philippe Mendels and Frederic Mila deals with such phases. You find the pdf in stud.ip.

a. Read the first pages (331-336) of the chapter. You should understand the introduced objects with the help of the lecture.

b. Show that the basis states (13.3.) are time reversal invariant.

c. What are the expectation values of $\underset{\sim}{s}^z$ with respect to the basis states (13.3.)?

d. Show that $\underset{\sim}{Q}^{\alpha\beta} = 0$ for $s = 1/2$.

e. Evaluate the commutators of $\underset{\sim}{Q}$ (13.6) with $\underset{\sim}{s}^z$? Consider the difference between the components of $\underset{\sim}{Q}$ (13.6) and the components of a spherical tensor of rank 2 as discussed in the lecture. The analogy is given by the difference between spherical harmonics and orbitals for $l = 2$.

f. If you want read more of the chapter.

13.2 Application of the Wigner-Eckart theorems for two interacting spins

Please use for the following problem the article R. Schnalle, J. Schnack, *Calculating the energy spectra of magnetic molecules: application of real- and spin-space symmetries*, Int. Rev. Phys. Chem. **29** (2010) 403-452 as a reference. You find the pdf in stud.ip.

a. According to the Wigner-Eckart theorem we have

$$\langle\, \alpha\, S\, M \,|\, \underset{\sim}{T}_q^{(k)} \,|\, \alpha'\, S'\, M' \,\rangle \quad = \quad (-1)^{S-M} \langle \alpha\, S || \underset{\sim}{\mathbf{T}}^{(k)} || \alpha'\, S' \rangle \begin{pmatrix} S & k & S' \\ -M & q & M' \end{pmatrix} \ . \tag{1}$$

Use this relation to evaluate the reduced matrix elements of

$$\langle s || \underset{\sim}{\mathbf{s}}^{(0)} || s \rangle \quad , \quad \langle s || \underset{\sim}{\mathbf{s}}^{(1)} || s \rangle \ . \tag{2}$$

b. In the following we want to evaluate the eigenvalues of a dot-product of two angular moments s_1 and s_2. Such terms appear e.g. in the Heisenberg model or in the spin-orbit interaction. For a spherical tensor, that is a product of two other spherical tensors, the reduced matrix element is

$$\langle s_1\, s_2\, S || \left\{ \underset{\sim}{\mathbf{T}}^{(k_1)} \otimes \underset{\sim}{\mathbf{T}}^{(k_2)} \right\}_q^{(k)} || s_1\, s_2\, S' \rangle \tag{3}$$

$$= \quad [(2S+1)\,(2S'+1)\,(2k+1)]^{\frac{1}{2}} \begin{pmatrix} s_1 & s_1' & k_1 \\ s_2 & s_2' & k_2 \\ S & S' & k \end{pmatrix}$$

$$\times \langle s_1 || \underset{\sim}{\mathbf{T}}^{(k_1)} || s_1 \rangle \langle s_2 || \underset{\sim}{\mathbf{T}}^{(k_2)} || s_2 \rangle \ , \tag{4}$$

compare Eq. (19) in the reference.

Calculate the eigenvalues of

$$\underset{\sim}{H} \quad = \quad -2J\, \underset{\sim}{\mathbf{s}}_1 \cdot \underset{\sim}{\mathbf{s}}_2 = 2\sqrt{3} J \left\{ \underset{\sim}{\mathbf{s}}_1^{(1)} \otimes \underset{\sim}{\mathbf{s}}_2^{(1)} \right\}^{(0)} \ , \tag{5}$$

which we know already from an earlier problem.

Learn about the various Wigner symbols from the literature.