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Problem sheet 6

6.1 Shift operator

Show that the operators

$$\underset{\sim}{T}_a \;=\; \mathrm{e}^{-\frac{i a \underset{\sim}{p}}{\hbar}} \ , \quad a \in \mathbb{R} \tag{1}$$

together with squential application form a group.
What are the properties of this group?

6.2 Cyclic groups

Consider the following groups

$$C_N \;=\; \left(\left\{\underset{\sim}{1} = \underset{\sim}{T}^0 = \underset{\sim}{T}^N, \underset{\sim}{T}, \underset{\sim}{T}^2, \dots, \underset{\sim}{T}^{N-1}\right\}, \cdot\right) \tag{2}$$

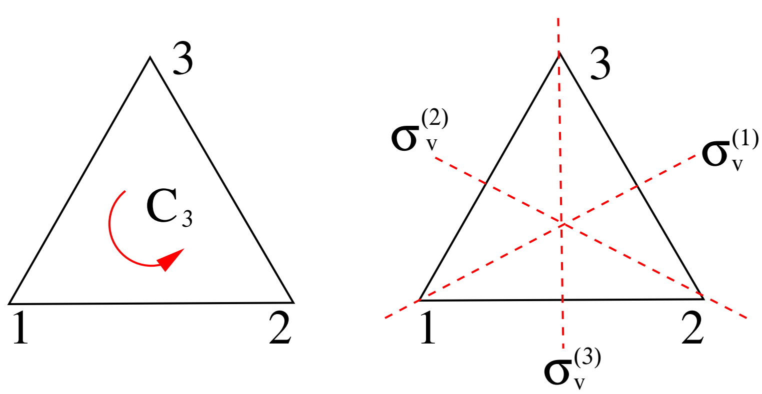
$$EW_N \;=\; \left(\left\{1, e^{-i\,\frac{2\pi}{N}}, e^{-i\,2\,\frac{2\pi}{N}}, \dots, e^{-i\,(N-1)\,\frac{2\pi}{N}}\right\}, \cdot\right) \tag{3}$$

$$Z_N \;=\; \left(\{0, 1, 2, \dots, N-1\}, + \quad \text{modulo} \quad N\right) \ . \tag{4}$$

Check whether these are indeed groups. What are their properties? Is there anything you noticed about these groups?

6.3 The group C_{3v}

The group C_{3v} consists of the symmetry transformations of an equilateral triangle. The group contains six elements: the rotations C_3 of 120° about the axis through the midpoint of and perpendicular to the triangle as well as reflections $\sigma_v^{(1)}, \sigma_v^{(2)}, \sigma_v^{(3)}$ about mirror planes spanned by the C_3 axis and the respective bisectrix.



As an example the following two operations are given

$$C_3(1,2,3) \;=\; (3,1,2) \qquad \text{and} \tag{5}$$
$$\sigma_v^{(2)}(1,2,3) \;=\; (3,2,1)\ . \tag{6}$$

a. Set up the group table.

b. Look for subgroups.

c. Please repeat the meaning of *conjugated* and *Conjugacy Classes* and determine the conjugacy classes for C_{3v}.

6.4 Dimensions of orthogonal subspaces

We consider a spin ring of N paramagnetic moments with spin quantum number s and identical nearest neighbor interaction J. The Hamiltonian is that of the Heisenberg model with periodic boundary conditions, i.e.,

$$\underset{\sim}{H} \;=\; -2J\sum_i \underset{\sim}{\vec{s}}_i \cdot \underset{\sim}{\vec{s}}_{i+1}\ , \qquad \text{with} \qquad N+1 \equiv 1\ , \forall i s_i = s\ . \tag{7}$$

The following product states form an othonormal basis set in Hilbert space

$$\underset{\sim}{s}_i^z \mid a_1, a_2, \dots, a_i, \dots, a_N \,\rangle \;=\; (s-a_i) \mid a_1, a_2, \dots, a_i, \dots, a_N \,\rangle\ , \tag{8}$$

with $a_i = s - m_i$.
In the following we fix $N=6$ and $s=3/2$.

a. Determine the dimension of the Hilbert space.

b. Dig out your Mathematica script of problem 4.2 and evaluate the diemensions of the orthogonal subspaces $\mathcal{H}(M)$. Since

$$\mathcal{H} = \bigoplus_{M=-M_{\max}}^{+M_{\max}} \mathcal{H}(M) \tag{9}$$

holds, the sum of these dimensionens must coincide with the dimension of $\mathcal{H}$.

c. Now arrange for each subspace$\mathcal{H}(M)$ the product state into cycles and evaluate the dimensions of the orthogonal subspaces $\mathcal{H}(M,k)$.

d. Although we did not use the full spin rotational symmetry, we can nevertheless calculate the dimensions of the orthogonal subspaces $\mathcal{H}(S,M,k)$. How?