Symmetric polynomials in physics

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Abstract. We give two examples where symmetric polynomials play an important rôle in physics: First, the partition functions of ideal quantum gases are closely related to certain symmetric polynomials, and a part of the corresponding theory has a thermodynamical interpretation. Further, the same symmetric polynomials also occur in Berezin's theory of quantization of phase spaces with constant curvature.

1. Introduction

It often happens that mathematical theories have unexpected applications in physics. In the present case of the theory of symmetric polynomials (SP) we have, additionally, the remarkable situation, that physicists have re-discovered certain fragments of the theory of SP in order to solve problems in few-particle quantum statistical mechanics [1],[2],[3],[4],[5],[6]. Actually, it turns out that a part of the physical theory of ideal quantum gases is equivalent to a part of the theory of SP if a certain translation scheme is applied, see below. The central idea of this scheme, namely that partition functions can be considered as evaluations of certain SP, is not novel, but appeared at various places in the literature, see [7],[8],[9],[10],[11], often in the context of generalized statistics. Nevertheless, the relevance for the problems treated in the above-mentioned articles and the consequences of this observation seem to have remained largely unnoticed.

A second field where SP might be important tools is the theory of quantization in the form suggested by F. A. Berezin [12] and subsequently further developed, see e. g. [13]. Here the same SP as in quantum statistical mechanics occur in the expansion of the quantization operator for two-dimensional phase spaces with constant curvature. One could speculate about the underlying reasons and possible extensions of this connection.

2. SP and ideal quantum gases

We will only explain the basic idea of the connection between SP and quantum statistical theory of ideal gases. Further details may be found in [14] and [15] and in the literature quoted there.

It is well-known that the eigenstates of the N-particle Hamiltonian without interactions can be characterized by "occupation number sequences" $i \mapsto n_i$. Here n_i is the occupation number of the i-th energy level E_i of the 1-particle Hamiltonian. Hence

$$\sum_i n_i = N, \tag{1}$$

and, for fermions, additionally

$$n_i \in \{0, 1\}. \tag{2}$$

Equivalently, each eigenstate can be characterized by a monomial of degree N

$$\boldsymbol{x}^{\boldsymbol{n}} \equiv \prod_i x_i^{n_i}, \tag{3}$$

where the x_i are abstract, commuting variables corresponding to the energy levels and $\boldsymbol{n}$ denotes the whole occupation number sequence. The energy eigenvalues corresponding to these eigenstates are

$$E = \sum_i E_i n_i. \tag{4}$$

Now we can express the N-particle partition function as

$$Z_N^{\pm}(\beta) = \sum_E e^{-\beta E} \;=\; \sum_{\mathbf{n}} \exp\left(-\beta \sum_i E_i n_i\right) \tag{5}$$

$$= \sum_{\mathbf{n}} \prod_i \left(e^{-\beta E_i}\right)^{n_i} \;=\; \underbrace{\sum_{\mathbf{n}} \prod_i x_i^{n_i}}\Bigg|_{x_i = \exp(-\beta E_i)} \tag{6}$$

$$\equiv \begin{cases} b_N(x_1, x_2, \ldots) & : \text{Bosons}(+) \\ f_N(x_1, x_2, \ldots) & : \text{Fermions}(-) \end{cases} \tag{7}$$

Here the sum over $\boldsymbol{n}$ is subject to the constraint (1) for bosons, and to (1), (2) for fermions. Since these constraints are invariant under permutations of the variables $x_1, x_2, \ldots$, the resulting sum of the monomials $\boldsymbol{x}^{\boldsymbol{n}}$ in (6) will be a symmetric polynomial of the $x_1, x_2, \ldots$. We call these SP "fermi polynomials " f_N or "bose polynomials" b_N, respectively. In the theory of SP the f_N are called "elementary SP" and the b_N "complete SP". However, in this article we will stick to our more physical nomenclature.

The partition function $Z_N(\beta)$ of a particular system is obtained by evaluation of the corresponding SP along the curve $\beta \mapsto x_i(\beta) = \exp(-\beta E_i)$, see figure 1. Hence we have a $1:1$ correspondence between certain SP and certain "partition types" of ideal gases. Here the "partition type" of a system is given by the number N of particles, the number $L \in \{1, 2, \ldots, \infty\}$ of abstract energy levels (or the dimension of the 1-particle Hilbert space) and the type of the statistics, Bose or Fermi. The values E_i of the energy levels, including their degeneracy, only determine the system and its particular partition function. Hence a SP corresponds not to a single system but to a large class of systems. It is then obvious, that mathematical relations between the f_N and the b_N can be translated into physical relations between the corresponding partition functions, irrespective of the values of the E_i.

There is a third kind of SP with a physical meaning in the theory of ideal gases, the "power sums"

$$p_n \equiv \sum_i x_i^n. \tag{8}$$

Evaluation at $x_i = \exp(-\beta E_i)$ gives

$$p_n|_{x_i = \exp(-\beta E_i)} = \sum_i \exp(-n\beta E_i) = Z_1(n\beta). \tag{9}$$

One of the central results of the elementary theory of SP is that each of the above families of SP, the f_N, the b_N, and the p_n, can be used as a "basis" of SP, in the sense that any SP can be expressed as a polynomial of the f_N (resp. b_N or p_n). This implies that the fermionic partition functions can be expressed by means of the bosonic ones and vice versa. Moreover,

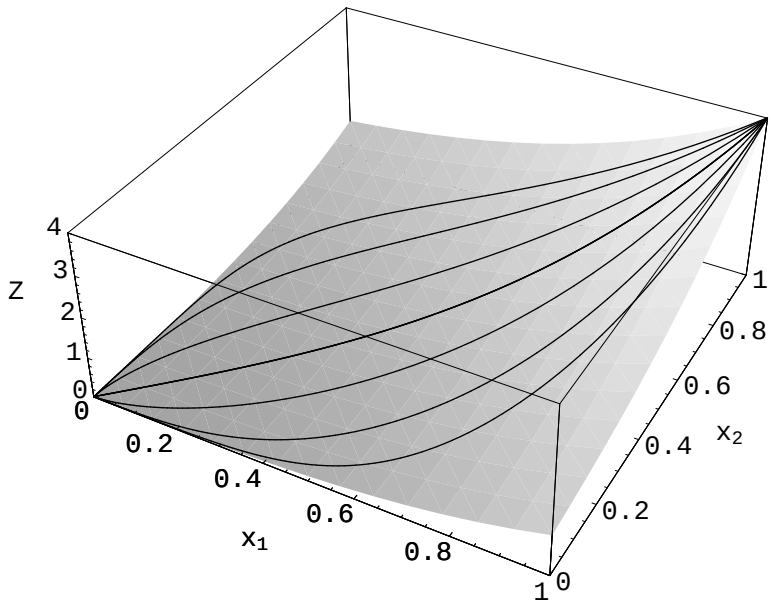

Figure 1. This figure shows the graph of the bose ploynomial $b_3 = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$ with $N = 3$ and $L = 2$. A selected number of curves parametrized by the inverse temperature β is shown which illustrate the partition functions for special systems obtained by the evaluation of b_3 at the values $x_1(\beta) = \exp(-\beta E_1)$ and $x_2(\beta) = \exp(-\beta E_2)$.

both in turn can be expressed by means of the 1-particle partition function at different inverse temperatures $n\beta$. These relations are more or less known in the physical literature but their origin in the theory of SP has only recently be disclosed [14].

SP can be defined through their generating functions. In statistical mechanics the corresponding generationg functions are called "grand canonical partitions functions" and will be denoted by $B(z)$ for bosons and $F(z)$ for fermions. The formal parameter z is physically interpreted as the fugacity $z = \exp(-\beta\mu)$, where μ is the chemical potential. The physical domain of z is $(0, 1)$ for bosons and $(0, \infty)$ for fermions.

There exists a fundamental symmetry $\omega : \Lambda \longrightarrow \Lambda$ of the ring Λ of SP, which maps the f_N onto the b_N, see [16]. It is connected with the equation $F(z)B(-z) = 1$. The physical interpretation of this relation is that the fermionic grand canonical partitions function is related to the analytical continuation of the bosonic one to negative z and vice versa. Similar relations for other thermodynamic functions are implied. Whether analytical continuation is possible depends on the system under consideration. For the system of particles in a box it has been shown that analytical continuation of the partition functions is possible in the thermodynamic limit [3] and the resulting Bose-Fermi symmetry has been discussed. Another Bose-Fermi symmetry has been observed for particles in odd space dimensions confined by a common harmonic oscillator potential [5]. Here $Z_1(\beta) = -Z_1(-\beta)$ implies $Z_N^+(\beta) = (-1)^N Z_N^-(-\beta)$. Since here analytical continuation is involved w. r. t. the β-plane, it is not yet clear how this symmetry is related to that given by ω

These and further relations and translation schemes will be sketched in the table below. For details see [14] and, as a standard reference for the theory of SP, [16].

Physics	*Mathematics*
Abstract energy levels	Variables $x_1, x_2, \ldots$
Occupation by N particles	Monomials $\mathbf{x^n}$
Partition types	Symmetric polynomials $p(\mathbf{x}) = \sum_{\mathbf{n}} \mathbf{x^n}$
-	Ring of symmetric polynomials Λ
Partition function $Z_N(\beta)$	Evaluation of $p(\mathbf{x})$ at $x_i = e^{-\beta E_i},\ i = 1, \ldots, L$
$\left.\begin{array}{l} F(z) \\ B(z) \end{array}\right\}$ Grand canonical partition functions	$\left.\begin{array}{l} F(z) \\ B(z) \end{array}\right\}$ Generating functions for $f_n,\ b_n$
Fugacity $z = e^{-\beta\mu}$	$F(z) \equiv \prod_{i=1}^{L} (1 + x_i\, z) = \sum_{n=0}^{L} f_n\, z^n$ $B(z) \equiv \prod_{i=1}^{L} \frac{1}{1-x_i\, z} = \sum_{n=0}^{L} b_n\, z^n$
Bose- Fermi symmetry by analytical continuation in the z-plane [3]	$B(z)F(-z) = 1$
	$\left.\begin{array}{l} \omega : \Lambda \longrightarrow \Lambda \\ \omega : f_n \mapsto b_n \end{array}\right\}$ involutive automorphism of graded rings
$\sum_i \left(e^{-\beta E_i}\right)^n = Z_1(n\beta)$	p_n, evaluated
-	Generating function: $P(z) = \frac{d}{dz} \log B(z) = \sum_{n=1}^{L} p_n z^n$
$\langle N \rangle$	$zP(z)$
Landsberg's identities (Appendix E of his 1961 textbook, rediscovered several times)	Newton's identity: $n f_n = \sum_{r=1}^{n} (-1)^{r-1} p_r f_{n-r}$

3. SP and Berezin quantization

The main idea of Berezin's approach to quantization is the use of "generalized coherent states" (GCS) in order to establish an approximate equivalence between classical and quantum observables and states. GCS will be denoted by $|\alpha\rangle$ where the "parameter" α runs through some phase space M. The most important property of GCS is the completeness relation

$$\mathbb{1} = \int_M |\alpha\rangle\langle\alpha| \, d\alpha. \tag{10}$$

CGS are used to map operators A onto functions on M by means of

$$j(A)(\alpha) \equiv \frac{\langle\alpha|A|\alpha\rangle}{\langle\alpha|\alpha\rangle}. \tag{11}$$

The adjoint map is given by

$$j^*(f) = \int_M f(\alpha)|\alpha\rangle\langle\alpha| \, d\alpha, \tag{12}$$

where we ignore all questions about the exact domains of definition of (11) and (12). The "quantization operator" $j \circ j^*$ maps functions onto functions. It is used for semiclassical expansions of physical quantities w. r. t. powers of h, where h is a formal parameter ("Planck's constant") on which the Hilbert space and the GCS depend.

In the case of a flat phase space $M = \mathbf{R}^2$, the GCS are chosen as the usual coherent states discovered by E. Schrödinger. Then

$$j \circ j^* = \exp(-h^2\Delta), \tag{13}$$

where Δ is the Laplace operator in $\mathbf{R}^2$, see e. g. [17]. There are at least two other cases where $j \circ j^*$ can be calculated: The Lobachevskij plane L and the 2-sphere S^2. Here $j \circ j^*$ is given in the form of an infinite product, see [12], which resembles the grand canonical partition functions considered in the previous section. Hence $j \circ j^*$ can be expanded into power series w. r. t. Δ, where the coefficients are bose polynomials for L and fermi polynomial for S^2. Δ is the corresponding Laplace-Beltrami operator of L, resp. S^2. More explicitely:

Case 1

$$\begin{aligned}
M &= \text{LOBACHEVSKIJ plane } L \\
|\alpha\rangle &: \text{BEREZIN'S coherent states [12]} \\
j \circ j^* &= \prod_{n=0}^{\infty} \left(1 - \frac{h^2}{(1+nh)(1-(n-1)h)}\Delta\right)^{-1} && (14) \\
&= \sum_{N=0}^{\infty} b_N(x_1, x_2, \ldots)\Delta^N && (15) \\
\text{where} \quad x_n &= \frac{h^2}{(1+nh)(1+(n-1)h)} && (16)
\end{aligned}$$

Case2

$$\begin{aligned}
M &= \text{2-sphere } S^2 \\
|\alpha\rangle &: \text{BLOCH'S coherent states [17]} \\
h &= \frac{1}{2s} \quad (s: \text{ spin quantum number}) && (17)
\end{aligned}$$

$$j \circ j^* = \prod_{n=0}^{\infty} \left(1 + \frac{h^2}{(1+nh)(1+(n+1)h)}\Delta\right) \tag{18}$$

$$= \sum_{N=0}^{\infty} f_N(x_1, x_2, \ldots)\Delta^N \tag{19}$$

$$\text{where} \quad x_n = \frac{1}{(2s+n)(2s+n+1)} \tag{20}$$

It is remarkable, that the bose and fermi polynomials occur for the two simplest cases of 2-dimensional phase spaces with constant negative and positive curvature. Both kind of polynomials can be considered as special cases of the so-called Schur polynomials s_λ, where λ denotes a Young diagram or, equivalently, a partition of N, see [16]. In fact, $b_N = s_{(N)}$ and $f_N = s_{(1^N)}$, where (N) (resp. (1^N)) is the Young diagram consisting of a single row (resp. column). One might wonder whether also other Schur polynomials occur in the expansion of the quantization operator for other choices of phase spaces and GCS.

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