Quantum phase transitions in the exactly solved spin-1/2 Heisenberg-Ising two-leg ladder

Taras Verkholyak[1] and Jozef Strečka[2]

[1]Institute for Condensed Matter Physics, National Academy of Sciences of Ukraine,
1 Svientsitskii Street, L'viv-11, 79011, Ukraine
[2]Department of Theoretical Physics and Astrophysics, Faculty of Science,
P. J. Šafárik University, Park Angelinum 9, 040 01 Košice, Slovak Republic

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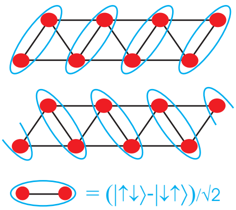
Outline

Introduction

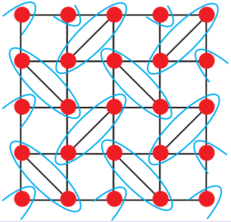
- Mutual interplay between spin frustration and quantum fluctuations in frustrated quantum spin models generally leads to
  - exotic ground states
  - fractional magnetization plateaux
  - enhanced magnetocaloric effect
- Low-temperature properties of frustrated quantum spin models are far from being fully understood yet ⇒ it is desirable to search for

exactly solvable frustrated quantum spin models

Majumdar-Ghosh model



Shastry-Sutherland model



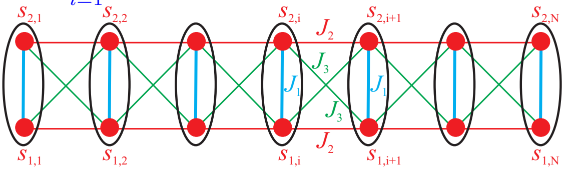
- ground state(s)
- low-lying excitations
- special constraints on interactions
- factorization/disentanglement

[1] H.T. Diep, *Frustrated Spin Systems*, World Scientific, Singapore, 2004.
[2] U. Schollwöck, J. Richter, D.J.J. Farnell, R.F. Bishop, *Quantum Magnetism*, Springer, Berlin, 2004.
[3] C. Lacroix, P. Mendels, F. Mila, *Introduction to Frustrated Magnetism*, Springer, Berlin, 2011.

Theoretical motivation

The spin-1/2 quantum Heisenberg model on two-leg ladder:

$$H = \sum_{i=1}^{N} [J_1 \mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i} + J_2(\mathbf{s}_{1,i} \cdot \mathbf{s}_{1,i+1} + \mathbf{s}_{2,i} \cdot \mathbf{s}_{2,i+1}) + J_3(\mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i+1} + \mathbf{s}_{2,i} \cdot \mathbf{s}_{1,i+1})]$$



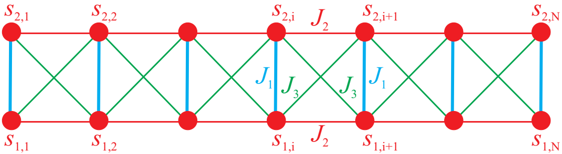
- J_1–intra-rung interaction
- J_2–intra-leg interaction
- J_3–crossing interaction

- intermediate in between one- and two-dimensions
- generally not integrable (exactly tractable)
- exact rung singlet-dimer state for $J_2 = J_3$ and $J_2/J_1 \leq 0.7135$ [Gelfand, PRB **43**, 8644 (1991); Xian, PRB **52**, 12485 (1995)].
- extensive studies (DMRG, ED, series, bosonization, etc.)
- field-induced Luttinger liquid with gapless excitations
- magnetization plateau [Honecker, Mila, Troyer, EPJB **15**, 227 (2000)].

Experimental motivation

The spin-1/2 quantum Heisenberg model on two-leg ladder:

$$H = \sum_{i=1}^{N} [J_1 \mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i} + J_2(\mathbf{s}_{1,i} \cdot \mathbf{s}_{1,i+1} + \mathbf{s}_{2,i} \cdot \mathbf{s}_{2,i+1}) + J_3(\mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i+1} + \mathbf{s}_{2,i} \cdot \mathbf{s}_{1,i+1})]$$

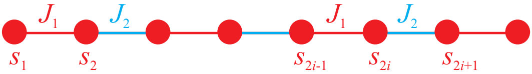


- J_1–intra-rung interaction
- J_2–intra-leg interaction
- J_3–crossing interaction

- two-leg ladder cuprates with dominating intra-rung coupling
  - $Cu_2(C_5H_{12}N_2)_2Cl_4$
  - $(C_5H_{12}N)_2CuBr_4$
  - $(5IAP)_2CuBr_4 \cdot 2H_2O$
- two-leg ladder cuprates with dominating intra-leg coupling
  - $KCuCl_3$
  - NH_4CuCl_3
  - $KCuBr_3$

Heisenberg-Ising (Ising-Heisenberg) models

Heisenberg-Ising chain: Lieb, Schultz, Mattis, Ann. Phys. **16**, 407 (1961)

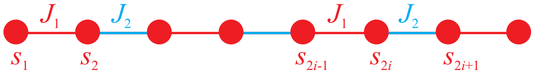


- J_1–Heisenberg interaction
- J_2–Ising interaction

$$H = \sum_{i=1}^{N} \left[J_1 (s_{2i-1}^x s_{2i}^x + s_{2i-1}^y s_{2i}^y + \Delta s_{1,i}^z s_{2,i}^z) + J_2 s_{2i}^z s_{2i+1}^z \right]$$

Heisenberg-Ising (Ising-Heisenberg) models

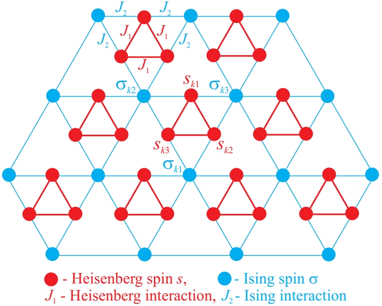
Heisenberg-Ising chain: Lieb, Schultz, Mattis, Ann. Phys. **16**, 407 (1961)



- J_1–Heisenberg interaction
- J_2–Ising interaction

$$H = \sum_{i=1}^{N} \left[J_1 (s^x_{2i-1} s^x_{2i} + s^y_{2i-1} s^y_{2i} + \Delta s^z_{1,i} s^z_{2,i}) + J_2 s^z_{2i} s^z_{2i+1} \right]$$

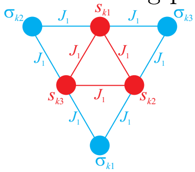
Exactly solvable Heisenberg-Ising (Ising-Heisenberg) models:



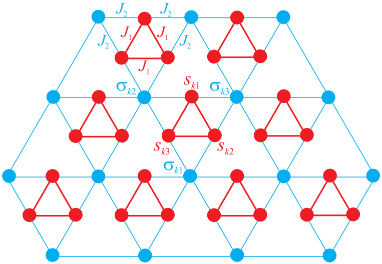
$$H = J_1 \sum_{(i,j)} [s^x_i s^x_j + s^y_i s^y_j + \Delta s^z_i s^z_j] + J_2 \sum_{(k,l)} s^z_k \sigma^z_l$$

Hamiltonian as a sum of its commuting parts:

$$H = \sum_k H_k$$



Heisenberg-Ising (Ising-Heisenberg) models



● - Heisenberg spin s, ● - Ising spin σ
J_1 - Heisenberg interaction, J_2 - Ising interaction

$$H = J_1 \sum_{(i,j)} [s_i^x s_j^x + s_i^y s_j^y + \Delta s_i^z s_j^z] + J_2 \sum_{(k,l)} s_k^z \sigma_l^z$$

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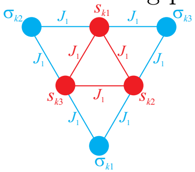


Partition function:

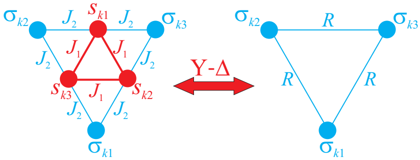
$$Z = \sum_{\{\sigma_k\}} \prod_k \mathrm{Tr}_k \exp(-\beta H_k)$$

Generalized star-triangle transformation:

$$\mathrm{Tr}_k \exp(-\beta H_k) = A \exp[\beta R(\sigma_{k1}^z \sigma_{k2}^z + \sigma_{k2}^z \sigma_{k3}^z + \sigma_{k3}^z \sigma_{k1}^z)]$$

Mapping equivalence:

$$Z = A^{N_c} Z_{\mathrm{IM}}(\beta, R)$$



Poster: J. Čisárová, J.S., F. Mila

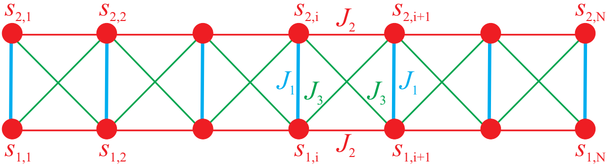
Spin-1/2 Heisenberg-Ising two-leg ladder

Hamiltonian:

$$H = \sum_{i=1}^{N} \Big[J_1(\mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i})_\Delta + J_2(s^z_{1,i}s^z_{1,i+1} + s^z_{2,i}s^z_{2,i+1}) + J_3(s^z_{2,i}s^z_{1,i+1} + s^z_{1,i}s^z_{2,i+1}) - h(s^z_{1,i} + s^z_{2,i}) \Big]$$

$$(\mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i})_\Delta = s^x_{1,i}s^x_{2,i} + s^y_{1,i}s^y_{2,i} + \Delta s^z_{1,i}s^z_{2,i}$$

- $J_1(\Delta)$ – XXZ Heisenberg intra-rung interaction
- J_2 – Ising intra-leg interaction
- J_3 – Ising crossing interaction
- h – external magnetic field



Bond representation

z-projection $S_i^z = s_{1,i}^z + s_{2,i}^z$ commutes with Hamiltonian $[S_i^z, H] = 0$:

$$|\phi_{0,0}^i\rangle = \frac{1}{\sqrt{2}}(|\downarrow_{1,i}\uparrow_{2,i}\rangle - |\uparrow_{1,i}\downarrow_{2,i}\rangle), \qquad |\phi_{1,0}^i\rangle = \frac{1}{\sqrt{2}}(|\downarrow_{1,i}\uparrow_{2,i}\rangle + |\uparrow_{1,i}\downarrow_{2,i}\rangle),$$

$$|\phi_{-}^i\rangle = \frac{1}{\sqrt{2}}(|\uparrow_{1,i}\uparrow_{2,i}\rangle - |\downarrow_{1,i}\downarrow_{2,i}\rangle), \qquad |\phi_{+}^i\rangle = \frac{1}{\sqrt{2}}(|\uparrow_{1,i}\uparrow_{2,i}\rangle + |\downarrow_{1,i}\downarrow_{2,i}\rangle).$$

The action of spin operators on the bond basis:
I. scalar product from XXZ Heisenberg interaction:

$$(\mathbf{s}_{1,i}\cdot\mathbf{s}_{2,i})_\Delta|\phi_{0,0}^i\rangle = -\frac{2+\Delta}{4}|\phi_{0,0}^i\rangle, \qquad (\mathbf{s}_{1,i}\cdot\mathbf{s}_{2,i})_\Delta|\phi_{1,0}^i\rangle = \frac{2-\Delta}{4}|\phi_{1,0}^i\rangle,$$

$$(\mathbf{s}_{1,i}\cdot\mathbf{s}_{2,i})_\Delta|\phi_{+}^i\rangle = \frac{\Delta}{4}|\phi_{+}^i\rangle, \qquad (\mathbf{s}_{1,i}\cdot\mathbf{s}_{2,i})_\Delta|\phi_{-}^i\rangle = \frac{\Delta}{4}|\phi_{-}^i\rangle,$$

II. s^z-operators involved in Ising interactions:

$$s_{1,i}^z|\phi_{0,0}^i\rangle = -\frac{1}{2}|\phi_{1,0}^i\rangle,\ s_{2,i}^z|\phi_{0,0}^i\rangle = \frac{1}{2}|\phi_{1,0}^i\rangle, \quad s_{1,i}^z|\phi_{+}^i\rangle = \frac{1}{2}|\phi_{-}^i\rangle,\ s_{2,i}^z|\phi_{+}^i\rangle = \frac{1}{2}|\phi_{-}^i\rangle,$$

$$s_{1,i}^z|\phi_{1,0}^i\rangle = -\frac{1}{2}|\phi_{0,0}^i\rangle,\ s_{2,i}^z|\phi_{1,0}^i\rangle = \frac{1}{2}|\phi_{0,0}^i\rangle, \quad s_{1,i}^z|\phi_{-}^i\rangle = \frac{1}{2}|\phi_{+}^i\rangle,\ s_{2,i}^z|\phi_{-}^i\rangle = \frac{1}{2}|\phi_{+}^i\rangle.$$

Pseudospin representation

i) the "pure" subspace $n_i = 0$ for $|\phi^i_{0,0}\rangle = |\downarrow\rangle^i_0$ and $|\phi^i_{1,0}\rangle = |\uparrow\rangle^i_0$:

$$s^z_{1,i} = -\frac{1}{2}(a^+_i + a_i) = -\tilde{s}^x_i, \qquad s^z_{2,i} = \frac{1}{2}(a^+_i + a_i) = \tilde{s}^x_i,$$

$$(\mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i})_\Delta = a^+_i a_i - \frac{2+\Delta}{4} = \tilde{s}^z_i - \frac{\Delta}{4}.$$

ii) the "impurity" subspace $n_i = 1$ for $|\phi^i_{1,-}\rangle = |\downarrow\rangle^i_1$ and $|\phi^i_{1,+}\rangle = |\uparrow\rangle^i_1$:

$$s^z_{1,i} = \frac{1}{2}(a^+_i + a_i) = \tilde{s}^x_i, \quad s^z_{2,i} = \frac{1}{2}(a^+_i + a_i) = \tilde{s}^x_i, \quad (\mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i})_\Delta = \frac{\Delta}{4}.$$

General form of spin operators in the pseudospin representation:

$$s^z_{1,i} = (2n_i - 1)\tilde{s}^x_i, \quad s^z_{2,i} = \tilde{s}^x_i, \quad (\mathbf{s}_{1,i} \cdot \mathbf{s}_{2,i})_\Delta = (1 - n_i)\tilde{s}^z_i + \frac{\Delta}{4}(2n_i - 1)$$

The Hamiltonian in the pseudospin representation:

$$H = \sum_{i=1}^{N} \Big\{ [4J_2 n_i n_{i+1} - 2(J_2 - J_3)(n_i + n_{i+1} - 1)]\tilde{s}^x_i \tilde{s}^x_{i+1} + \frac{J_1\Delta}{4}(2n_i - 1) + J_1(1 - n_i)\tilde{s}^z_i - 2h n_i \tilde{s}^x_i \Big\}.$$

Unitary transformation method

Unitary transformation directly applied to the original Hamiltonian:

$$U = \prod_{i=1}^{N} \exp\left[-i\frac{\pi}{2}(s_{1,i}^x + s_{2,i}^x)\right] \exp\left(i\pi s_{1,i}^x s_{2,i}^x\right) \exp\left(-i\frac{\pi}{2} s_{2,i}^y\right) \exp\left(i\pi s_{2,i}^z\right)$$

yields the Hamiltonian of transverse Ising chain with 'composite' spins:

$$UHU^+ = \sum_{i=1}^{N} \Big\{ \left[J_2(4s_{1,i}^z s_{1,i+1}^z + 1) + 2J_3(s_{1,i+1}^z + s_{1,i}^z)\right] s_{2,i}^x s_{2,i+1}^x + J_1\left(\frac{1}{2} - s_{1,i}^z\right) s_{2,i}^z + \frac{J_1\Delta}{2} s_{1,i}^z - h(1 + 2s_{1,i}^z) s_{2,i}^x \Big\}.$$

The straightforward mapping relationship:

$$s_{1,i}^z = n_i - \frac{1}{2} \quad , \quad s_{2,i}^\alpha = \tilde{s}_i^\alpha$$

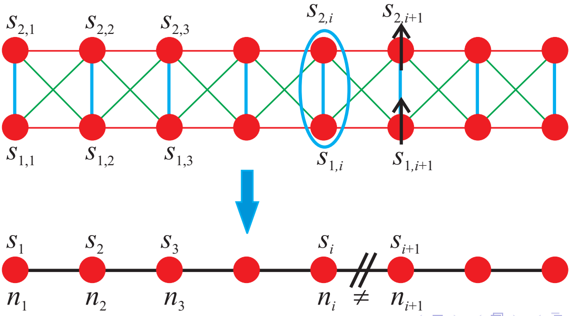
establishes an equivalence between the pseudospin Hamiltonian and the unitary-transformed Hamiltonian.

Composite quantum spin chain

The symmetrized form of the transformed Hamiltonian:

$$H = \sum_{i=1}^{N} \Big\{ \Big[2(J_2 + J_3) n_i n_{i+1} + 2(J_2 - J_3)(1 - n_i)(1 - n_{i+1}) \Big] \tilde{s}_i^x \tilde{s}_{i+1}^x + J_1 (1 - n_i) \tilde{s}_i^z - 2h n_i \tilde{s}_i^x + \frac{J_1 \Delta}{4} (2n_i - 1) \Big\}.$$

If $n_i \neq n_{i+1}$, the composite spin-chain model splits into independent fragments at this bond domain [Bose, Gayen, PRB **48**, 10653 (1993)].



Uniform configurations

The symmetrized form of the transformed Hamiltonian:

$$H = \sum_{i=1}^{N} \Big\{ \Big[2(J_2 + J_3) n_i n_{i+1} + 2(J_2 - J_3)(1 - n_i)(1 - n_{i+1}) \Big] \tilde{s}_i^x \tilde{s}_{i+1}^x + J_1(1 - n_i)\tilde{s}_i^z - 2h n_i \tilde{s}_i^x + \frac{J_1 \Delta}{4}(2n_i - 1) \Big\}.$$

If $n_i \neq n_{i+1}$, the composite spin-chain model splits into independent fragments at this bond domain [Bose, Gayen, PRB **48**, 10653 (1993)].

Two "uniform" configurations without translationally broken symmetry:
i) the "pure" configuration with all $n_i = 0$:

$$H^0 = \sum_{i=1}^{N} \Big[2(J_2 - J_3) \tilde{s}_i^x \tilde{s}_{i+1}^x + J_1 (\tilde{s}_i^z - \frac{\Delta}{4}) \Big],$$

ii) the "impurity" configuration with all $n_i = 1$:

$$H^1 = \sum_{i=1}^{N} \Big[2(J_2 + J_3) \tilde{s}_i^x \tilde{s}_{i+1}^x - 2h \tilde{s}_i^x + \frac{J_1 \Delta}{4} \Big].$$

Ground-state energy

Uniform "pure" configuration ≡ the Ising chain in a transverse field:

$$e_0^0 = \lim_{N\to\infty} \frac{E_0^0(N)}{N} = -\frac{J_1\Delta}{4} - \frac{(J_1 + |J_2 - J_3|)}{\pi}\mathbf{E}(\sqrt{1-\gamma^2}),$$

where $\gamma = \frac{J_1 - |J_2 - J_3|}{J_1 + |J_2 - J_3|}$ and $\mathbf{E}(\kappa) = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \sqrt{1 - \kappa^2 \sin^2\theta}$. (2nd kind)

Uniform "impurity" configuration ≡ the Ising chain in a longitudinal field:

$$e_0^1 = \lim_{N\to\infty} \frac{E_0^1(N)}{N} = \begin{cases} \frac{J_1\Delta}{4} + \frac{J_2+J_3}{2} - |h|, & \text{if } |h| > (J_2 + J_3), \\ \frac{J_1\Delta}{4} - \frac{J_2+J_3}{2}, & \text{if } |h| \leq (J_2 + J_3). \end{cases}$$

The uniform pure and impurity configurations are always lower in energy than any random configuration in the zero-field case (sketch of proof):

$$E_0^0(N_1 + N_2) \leq E_0^0(N_1) + E_0^0(N_2)$$
$$E_0^1(N_1 + N_2) \leq E_0^1(N_1) + E_0^1(N_2)$$

Uniform configurations as the only possible ground states (proof)

Hamiltonian of two independent transverse Ising chains of size N_1 and N_2:

$$H^0(N_1,N_2) = 2(J_2-J_3)\sum_{i=1}^{N_1-1}\tilde{s}_i^x\tilde{s}_{i+1}^x + J_1\sum_{i=1}^{N_1}\tilde{s}_i^z + 2(J_2-J_3)\sum_{i=N_1+1}^{N_1+N_2-1}\tilde{s}_i^x\tilde{s}_{i+1}^x + J_1\sum_{i=N_1+1}^{N_1+N_2}\tilde{s}_i^z = H^0(N_1+N_2) - 2(J_2-J_3)\tilde{s}_{N_1}^x\tilde{s}_{N_1+1}^x \quad (1)$$

Let $E_0^0(N_1,N_2) = E_0^0(N_1)+E_0^0(N_2)$ and $|\psi_0^{N_1,N_2}\rangle = |\psi_0^{N_1}\rangle|\psi_0^{N_2}\rangle$ be the lowest eigenvalue and eigenstate of $H^0(N_1,N_2)$ and let $E_0^0(N_1+N_2)$ and $|\psi_0^{N_1+N_2}\rangle$ be the lowest eigenvalue and eigenstate of $H^0(N_1+N_2)$.

Then, it is straightforward to show that

$$E_0^0(N_1)+E_0^0(N_2) = \langle\psi_0^{N_1,N_2}|H^0(N_1+N_2)|\psi_0^{N_1,N_2}\rangle$$
$$-2(J_2-J_3)\langle\psi_0^{N_1}|\tilde{s}_{N_1}^x|\psi_0^{N_1}\rangle\langle\psi_0^{N_2}|\tilde{s}_{N_1+1}^x|\psi_0^{N_2}\rangle \geq E_0^0(N_1+N_2).$$

Here, we have used that $\langle\psi_0^{N_1}|\tilde{s}_{N_1}^x|\psi_0^{N_1}\rangle = 0$ for any finite chain and that the lowest mean value of the operator $H^0(N_1+N_2)$ is achieved in its ground state (variational principle).

The same property is valid for the Ising chain in zero longitudinal field:

$$E_0^1(N_1+N_2) = E_0^1(N_1)+E_0^1(N_2)+\frac{|J_2+J_3|}{2} \leq E_0^1(N_1)+E_0^1(N_2).$$

Ground-state phase transitions

Uniform "pure" configuration ≡ the Ising chain in a transverse field:

$$e_0^0 = \lim_{N\to\infty} \frac{E_0^0(N)}{N} = -\frac{J_1\Delta}{4} - \frac{(J_1 + |J_2 - J_3|)}{\pi}\mathbf{E}(\sqrt{1-\gamma^2}),$$

where $\gamma = \frac{J_1 - |J_2 - J_3|}{J_1 + |J_2 - J_3|}$ and $\mathbf{E}(\kappa) = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \sqrt{1 - \kappa^2 \sin^2\theta}$. (2nd kind)

Uniform "impurity" configuration ≡ the Ising chain in a longitudinal field:

$$e_0^1 = \lim_{N\to\infty} \frac{E_0^1(N)}{N} = \begin{cases} \frac{J_1\Delta}{4} + \frac{J_2+J_3}{2} - |h|, & \text{if } |h| > (J_2 + J_3), \\ \frac{J_1\Delta}{4} - \frac{J_2+J_3}{2}, & \text{if } |h| \leq (J_2 + J_3). \end{cases}$$

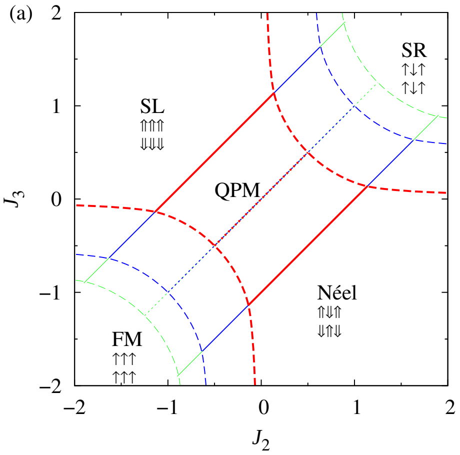
Ground-state phase transitions:

- First-order phase transitions at $e_0^0 = e_0^1$
- Second-order phase transitions at $|J_1| = |J_2 - J_3|$

Ground-state phases

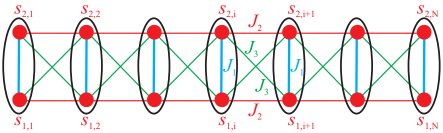
- **Quantum Paramagnetic (QPM) phase** appears if $e_0^0 < e_0^1$ and $|J_1| > |J_2 - J_3|$:
the absence of long-range order, the equivalent transverse Ising chain is in the disordered state ($\langle \tilde{s}_i^x \rangle = 0$). The rung singlet-dimer state dominates.
- **Stripe Leg (SL) phase** appears if $e_0^0 < e_0^1$ and $|J_1| < J_3 - J_2$:
the quantum long-range order, the equivalent transverse Ising chain is ordered.
$\langle s_{1,i}^z \rangle = \langle s_{1,i+1}^z \rangle = -\langle s_{2,i}^z \rangle = -\langle s_{2,i+1}^z \rangle = -\langle \tilde{s}_i^x \rangle \neq 0.$
The order parameter is $m^z = \frac{1}{2N} \sum_{i=1}^{N} (\langle s_{1,i}^z \rangle - \langle s_{2,i}^z \rangle) = \frac{1}{2}\left[1 - \frac{J_1^2}{(J_2 - J_3)^2}\right]^{\frac{1}{8}}$.
- **Néel phase** appears if $e_0^0 < e_0^1$ and $|J_1| < J_2 - J_3$:
the quantum long-range order, the equivalent transverse Ising chain is ordered.
$\langle s_{1,i}^z \rangle = -\langle s_{1,i+1}^z \rangle = -\langle s_{2,i}^z \rangle = \langle s_{2,i+1}^z \rangle = -\langle \tilde{s}_i^x \rangle \neq 0.$
The order parameter is $m^z = \frac{1}{2N} \sum_{i=1}^{N} (-1)^i (\langle s_{1,i}^z \rangle - \langle s_{2,i}^z \rangle) = \frac{1}{2}\left[1 - \frac{J_1^2}{(J_2 - J_3)^2}\right]^{\frac{1}{8}}$.
- **Stripe Rung (SR) phase** appears if $e_0^1 < e_0^0$ and $J_2 > 0$, $J_3 > 0$:
the classical ferromagnetic alignment along rungs accompanied with antiferromagnetic alignment along legs $\langle s_{1,i}^z \rangle = -\langle s_{1,i+1}^z \rangle = \langle s_{2,i}^z \rangle = -\langle s_{2,i+1}^z \rangle = 1/2$.
- **Ferromagnetic (FM) phase** appears if $e_0^1 < e_0^0$ and $J_2 < 0$, $J_3 < 0$:
This classical fully polarized spin state with $\langle s_{1,i}^z \rangle = \langle s_{2,i}^z \rangle = 1/2$.

Zero-field ground-state phase diagram



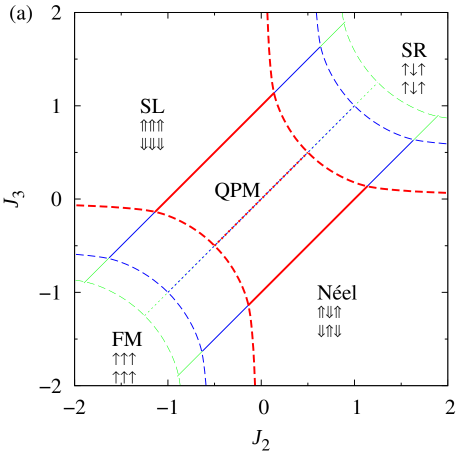
Quantum paramagnetic phase (QPM):

- region with the dominating J_1
- absence of long-range order
- dominating rung singlet dimers
- special case $J_2 = J_3$:
  - exact rung singlet dimers
  - no inter-rung correlations



Ground-state phase diagrams of the spin-1/2 Heisenberg-Ising ladder with $J_1 = 1$ and three different values of $\Delta =$ 0.0 (red), 1.0 (blue), 1.5 (green).

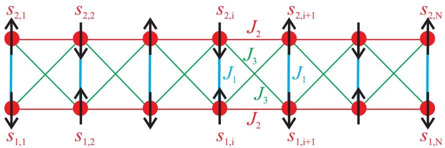
Zero-field ground-state phase diagram



Néel phase:

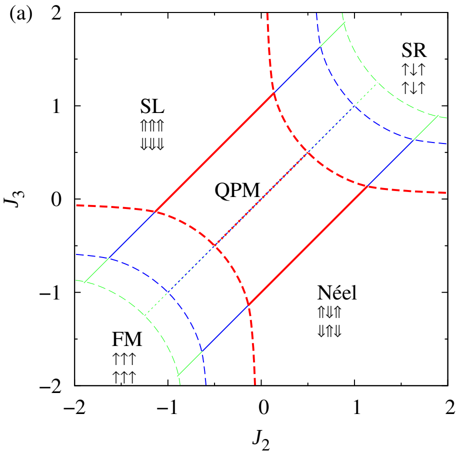
- region with J_2-AF and J_3-F
- second-order quantum phase transition at $|J_1| = J_2 - J_3$
- quantum reduction in staggered magnetization:

$$m = \frac{1}{2}\left[1 - \frac{J_1^2}{(J_2 - J_3)^2}\right]^{\frac{1}{8}}$$



Ground-state phase diagrams of the spin-1/2 Heisenberg-Ising ladder with $J_1 = 1$ and three different values of $\Delta =$ 0.0 (red), 1.0 (blue), 1.5 (green).

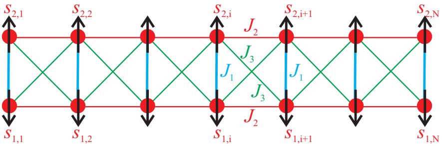
Zero-field ground-state phase diagram



Stripe Leg (SL) phase:

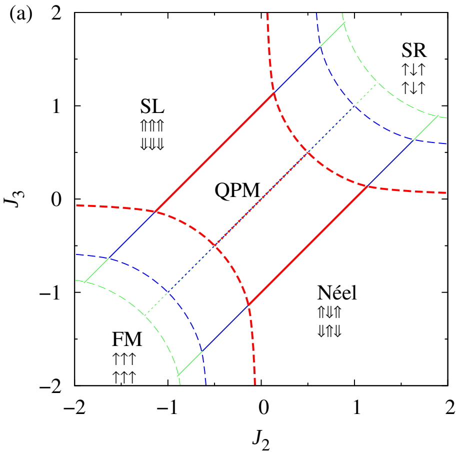
- region with J_2-F and J_3-AF
- second-order quantum phase transition at $|J_1| = J_3 - J_2$
- quantum reduction in staggered magnetization:

$$m = \frac{1}{2}\left[1 - \frac{J_1^2}{(J_2 - J_3)^2}\right]^{\frac{1}{8}}$$



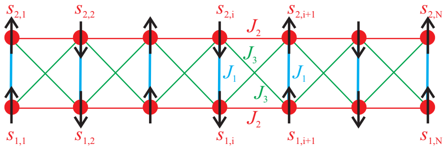
Ground-state phase diagrams of the spin-1/2 Heisenberg-Ising ladder with $J_1 = 1$ and three different values of $\Delta = 0.0$ (red), 1.0 (blue), 1.5 (green).

Zero-field ground-state phase diagram



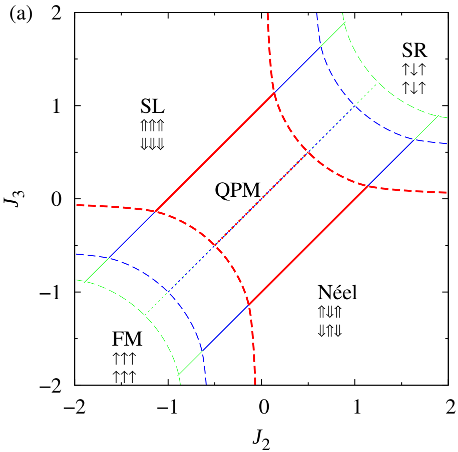
Stripe Rung (SR) phase:

- region with J_2-AF and J_3-AF
- first-order phase transition from QPM, Néel, SL
- classical long-range order
- fully saturated order parameter



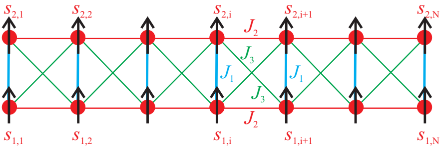
Ground-state phase diagrams of the spin-1/2 Heisenberg-Ising ladder with $J_1 = 1$ and three different values of $\Delta =$ 0.0 (red), 1.0 (blue), 1.5 (green).

Zero-field ground-state phase diagram



Ferromagnetic (FM) phase:

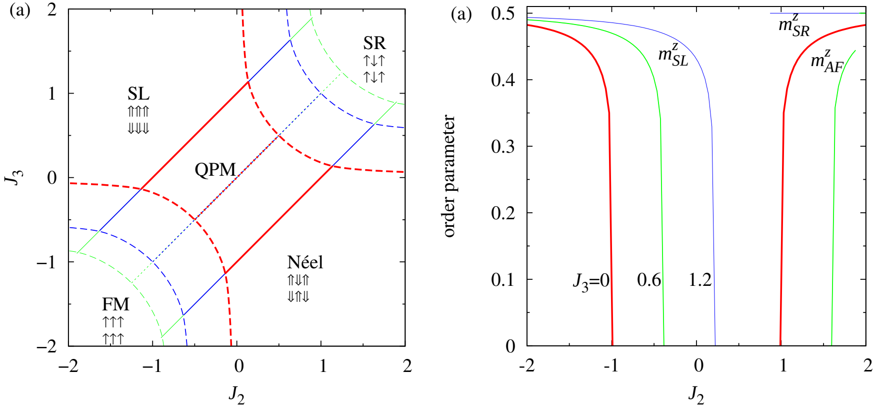
- region with J_2-F and J_3-F
- first-order phase transition from QPM, Néel, SL
- classical long-range order
- fully saturated order parameter



Ground-state phase diagrams of the spin-1/2 Heisenberg-Ising ladder with $J_1 = 1$ and three different values of Δ = 0.0 (red), 1.0 (blue), 1.5 (green).

Order parameters

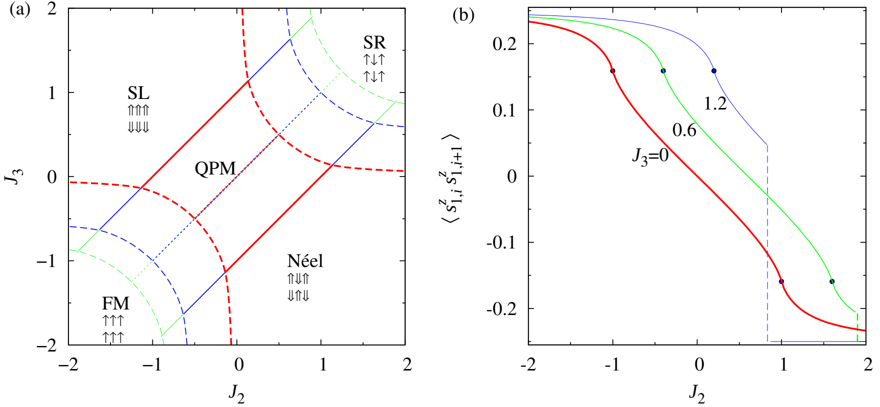
Staggered magnetization vs. intra-leg coupling J_2 for $J_1 = 1$ and $\Delta = 1$



- gradual disappearance of magnetization at continuous phase transitions
- abrupt jumps of magnetization at discontinuous phase transitions
- absence of long-range order in QPM

Nearest-neighbour spin-spin correlations

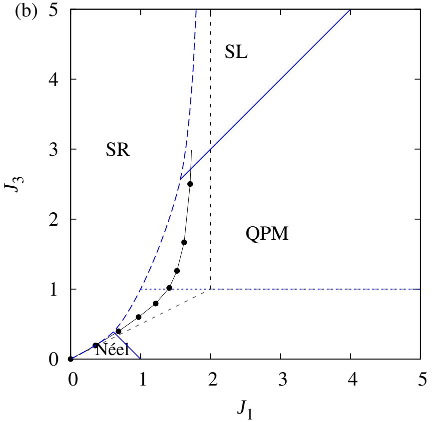
Intra-leg spin-spin correlation vs. intra-leg coupling J_2 for $J_1 = 1$ and $\Delta = 1$



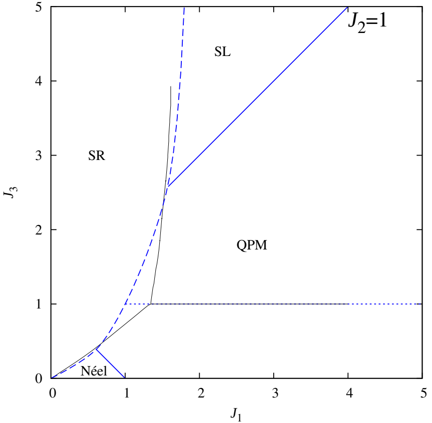
- weak singularity of spin-spin correlations at continuous phase transitions
- abrupt jumps of spin-spin correlations at discontinuous phase transitions
- rung singlet-dimer state dominates in QPM

Heisenberg, Ising and Heisenberg-Ising ladder ($J_2 = 1$)

Weihong *et al.*, PRB **57**, 11439 (1998)



Azzouz, Ramakko, CJP **86**, 509 (2008)



- blue lines - Heisenberg-Ising ladder with $\Delta = 1$
- black solid lines - Heisenberg ladder (series expansion and BMFT)
- black broken lines - Ising ladder $\Delta \to \infty$

Effect of non-zero external field

- uniform configurations need not be true ground states (proof)
- higher-period configurations cannot be ground states
- "staggered bond" (SB) phase as the only further ground state

The Hamiltonian:

$$H = \sum_{i=1}^{N} \Big\{ \Big[2(J_2 \; + \; J_3) n_i n_{i+1} + 2(J_2 - J_3)(1 - n_i)(1 - n_{i+1}) \Big] \tilde{s}_i^x \tilde{s}_{i+1}^x \\ + \; J_1 (1 - n_i) \tilde{s}_i^z - 2h n_i \tilde{s}_i^x + \frac{J_1 \Delta}{4} (2n_i - 1) \Big\}.$$

"Staggered bond" (SB) phase for $n_{2i-1} = 1$ and $n_{2i} = 0$ (or vice versa):

- alternation of "pure" and "impurity" states
- does not possess any "effective interaction" energy
- paramagnetic spins in alternating longitudinal and transverse field

Effect of non-zero external field

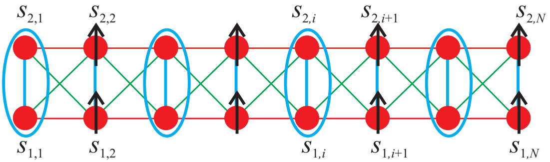
- uniform configurations need not be true ground states (proof)
- higher-period configurations cannot be ground states
- "staggered bond" (SB) phase as the only further ground state

The Hamiltonian:

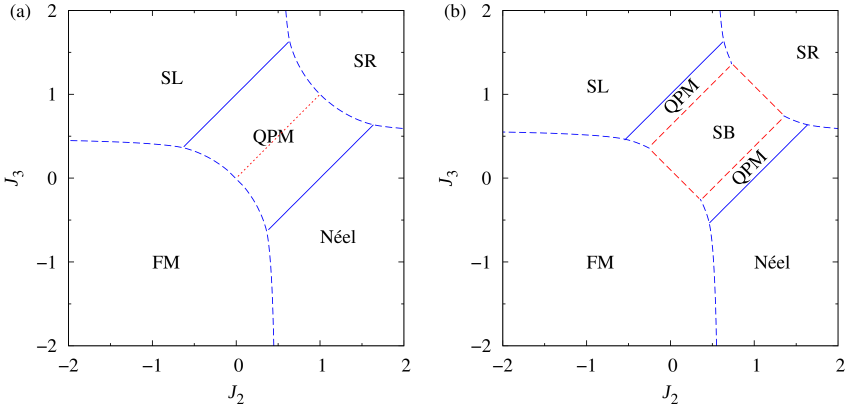
$$H = \sum_{i=1}^{N} \Big\{ \Big[2(J_2 + J_3) n_i n_{i+1} + 2(J_2 - J_3)(1-n_i)(1-n_{i+1}) \Big] \tilde{s}_i^x \tilde{s}_{i+1}^x + J_1(1-n_i)\tilde{s}_i^z - 2hn_i\tilde{s}_i^x + \frac{J_1\Delta}{4}(2n_i - 1) \Big\}.$$

"Staggered bond" (SB) phase for $n_{2i-1} = 1$ and $n_{2i} = 0$ (or vice versa):

- alternation of singlet dimers and fully polarized triplet ($\frac{1}{2}$-plateau)

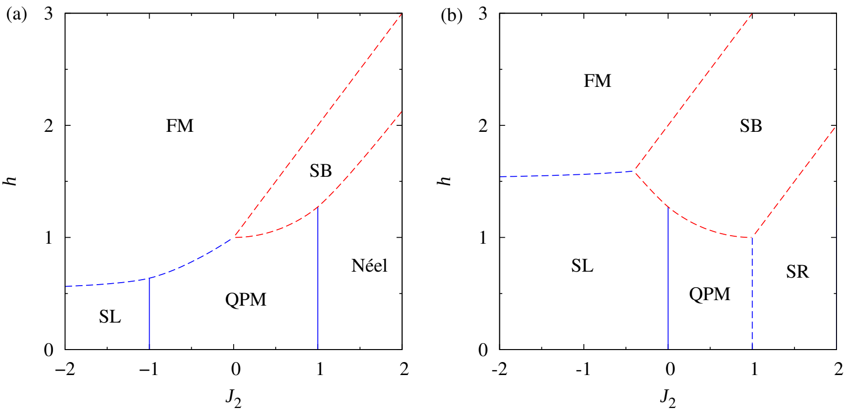


Ground-state phase diagram for $J_1 = 1$, $\Delta = 1$: (a) $h = 1.0$; (b) $h = 1.1$



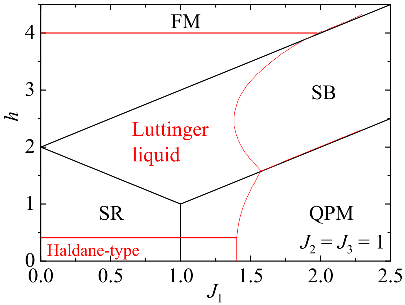
- topology of phase diagram remains the same for $h \leq 1$
- above $h \geq 1$ "staggered bond" (SB) phase appears along $J_2 = J_3$ line
- fractional magnetization plateau at one half of saturation magnetization

Ground-state phase diagram for $J_1 = 1$, $\Delta = 1$: (a) $J_3 = 0$; (b) $J_3 = 1$



- unfrustrated ladder: SB appears only for AF intra-leg interaction $J_2 > 0$
- frustrated ladder: SB appears also for F intra-leg interaction J_2
- frustration destabilizes Néel and QPM phases

Heisenberg and Heisenberg-Ising ladders ($J_2 = J_3 = 1$)



- red lines: phase boundaries of Heisenberg ladder according to Honecker, Mila, Troyer, EPJB **15**, 227 (2000)
- black lines: phase boundaries of the Heisenberg-Ising ladder with $\Delta = 1$
- only first-order phase transitions

- QPM reduces to exact rung singlet-dimer state
- SB - appearance of intermediate plateau
- identical magnetization process for $J_1 \gtrsim 1.5$
- quantum Haldane-type phase instead of classical SR phase
- gapless region with continously varying magnetization

Concluding remarks

- Exact results for the ground state of the spin-1/2 Heisenberg-Ising ladder
  - the exact solution has been found by two independent methods (pseudospin formulation, unitary transformation)
  - effect of the external magnetic field and the frustrating diagonal inter-leg Ising interaction has been investigated in particular
  - five different ground states (three quantum and two classical phases) have been found in a zero magnetic field
  - another peculiar SB phase, which has translationally broken symmetry, may occur when applying a non-zero external field
  - fractional magnetization plateau at half of the saturated magnetization has been consequently detected
- Future outlooks and open questions
  - thermodynamic and magnetic properties at non-zero temperatures
  - theoretical investigation of measures of the quantum entanglement
  - study of related Heisenberg-Ising models defined on sawtooth and dimer-plaquette chains, zig-zag ladder, Shastry-Sutherland lattice
  - perturbative approach for the Heisenberg two-leg ladder

More details can be found in:

T. Verkholyak, J. Strečka, preprint arXiv: 1204.1008

(submitted to J. Phys. A: Math. Theor.)

Thank you very much for you attention!